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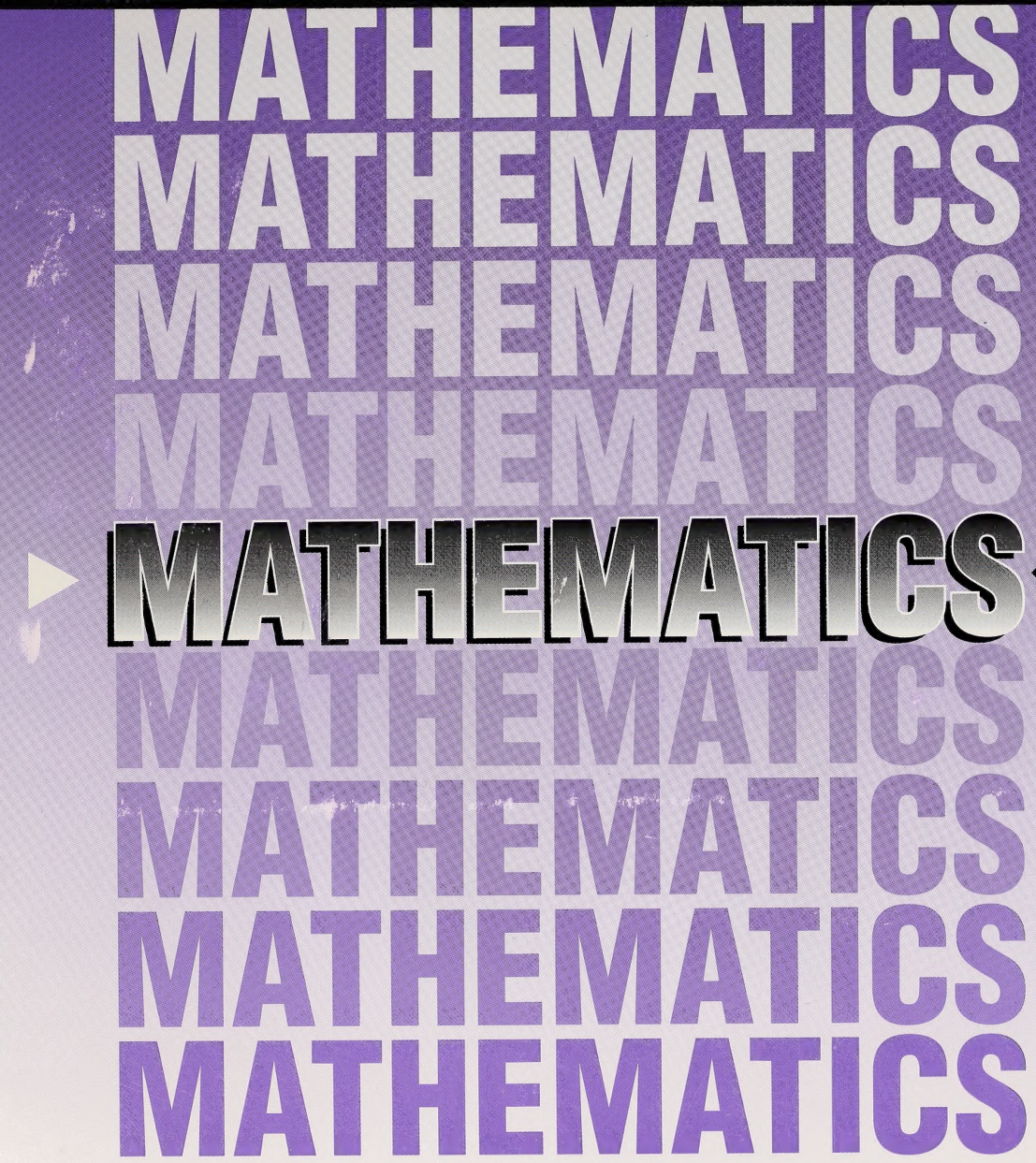
MODULE 3

MATHEMATICS 30

Exponential and Logarithmic Functions



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Mathematics 30

Module 3

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Distance Learning

Alberta
EDUCATION

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Student Module Booklet
Module 3
Exponential and Logarithmic Functions
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Welcome

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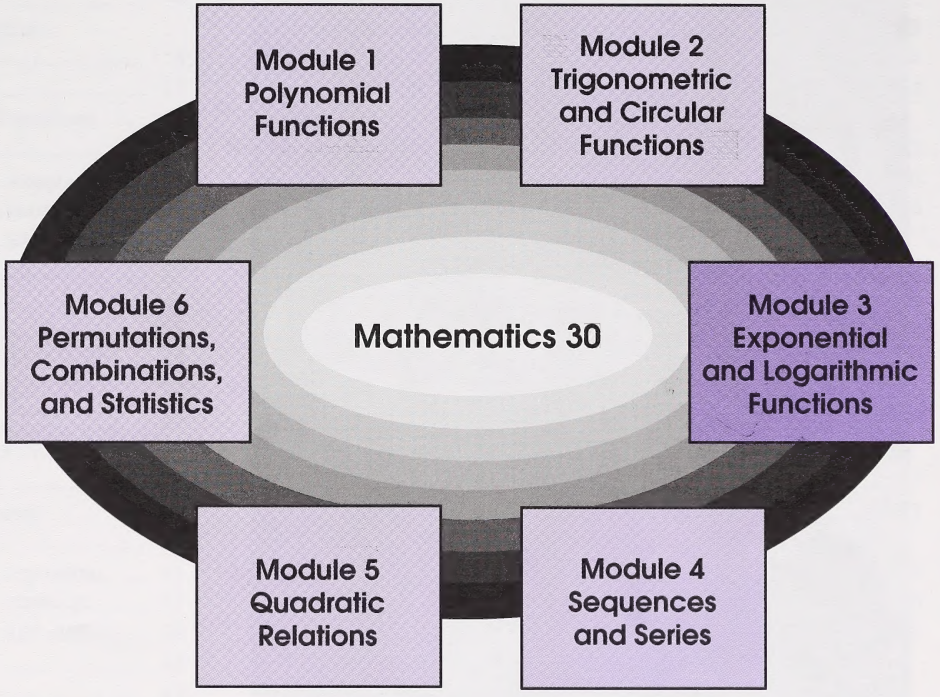
Welcome to Module 3. We hope you'll enjoy your study of Exponential and Logarithmic Functions.

Mathematics 30 contains six modules. Work through the modules in the order given, since several concepts build on each other as you progress in the course.



The document you are presently reading is called a Student Module Booklet. You may find visual cues or icons throughout it. Read the following explanations to discover what each icon prompts you to do.

- Use your graphing calculator.

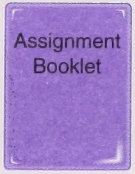
- Use your scientific calculator.

- Use computer software.

- Use the suggested answers in the Appendix to correct the activities.

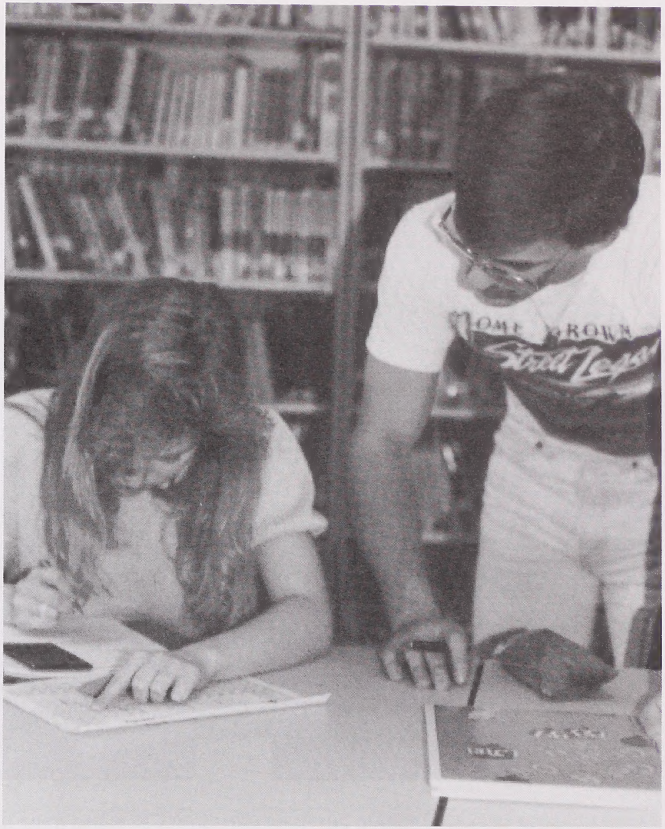
- View a videocassette.

- Pay close attention to important words or ideas.



- Answer the questions in the assignment booklet.

Note: Whenever the scientific calculator icon appears, you may use a graphing calculator instead.



There are no response spaces provided in this Student Module Booklet. This means that you will need to use your own paper for your responses. You should keep your response pages in a binder so that you can refer to them when you are reviewing or studying.

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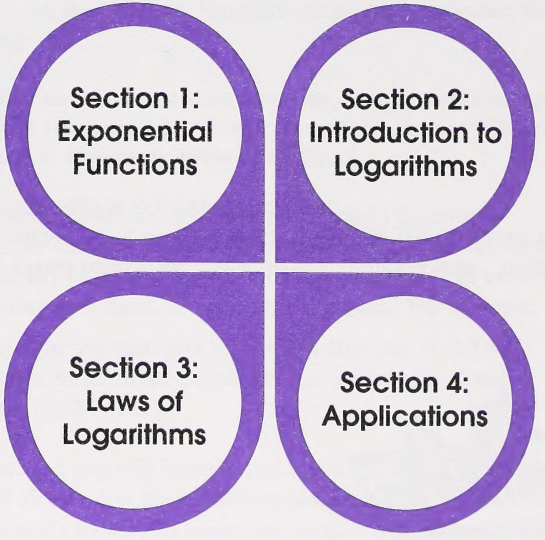
Module Overview

Before the advent of modern calculators and computers, how did people solve exponential equations $\left(\text{such as } 2^{x} = 25\right)$ or perform complicated and lengthy calculations $\left(\text{such as } 1.025^{100}\right)$? The development of logarithms in the early seventeenth century reduced the time and labour involved in complex calculations. For instance, a calculation like $\frac{(3.678)(489.7)}{491.6}$ could be done in seconds simply by adding and subtracting entries obtained from logarithm tables.

Logarithms were applied to problems in navigation, astronomy, engineering, and commerce. Today, logarithms are still essential tools. In the biological sciences, for instance, logarithms are used to model growth, from the growth of bacteria to the growth of animal and human populations.

In Section 1 you will work with exponential functions, reviewing the rules of exponents you encountered in previous courses. In Sections 2 and 3 you will be introduced to logarithmic functions, the rules that govern them, and how logarithmic functions relate to exponential functions. In the last section you will investigate a variety of applications of both exponential and logarithmic functions. Population growth, radioactive decay, compound interest, and decibel and Richter scales are a few of the everyday situations you will analyse.

Module 3
Exponential and Logarithmic Functions

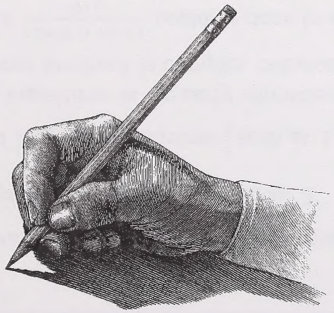

Section 1: Exponential Functions
Section 2: Introduction to Logarithms
Section 3: Laws of Logarithms
Section 4: Applications

Evaluation

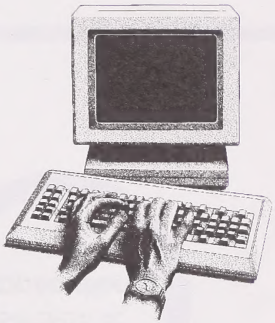
Your mark for this module will be determined by how well you complete the assignments at the end of each section and at the end of the module. In this module you must complete four section assignments and one final module assignment. The mark distribution is as follows:

Section 1 Assignment	**7 marks**
Section 2 Assignment	**18 marks**
Section 3 Assignment	**21 marks**
Section 4 Assignment	**24 marks**
Final Module Assignment	**30 marks**
TOTAL	**100 marks**

When doing the assignments, work slowly and carefully. You must do each assignment independently; but if you are having difficulties, you may review the appropriate section in this module booklet.



If you are working on a CML terminal, you will have a module test as well as a module assignment.



Note

There is a final supervised test at the end of this course. Your mark for the course will be determined by how well you do on the module assignments and the supervised final test.

Section 1: Exponential Functions

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Exponents commonly occur in the sciences (chemistry, biology, physics, and so on), business and economics, engineering, and architecture; in fact, in every area involving mathematics. Problems that incorporate the concept of exponents appear as early as 1650 B.C. At that time, the Egyptian scribe **Ahmes** penned the following:

> Each of seven peasants owns seven cats, each cat eats seven rats, each rat eats seven cobs of corn, and each cob of corn yields seven measures. How many measures are there?

A laboratory technician (as in the photograph) describing microscopic sizes, or an astronomer talking about the universe, automatically uses exponential notation—a convenient way of representing very small or very large numbers. For example, did you know that the smallpox virus has a diameter of 2.5×10^{-6} m? The modern notation for exponents was invented by **Rene Descartes** (1596–1650).

In this section you will identify and graph exponential functions. You will also review the laws of exponents. Finally, you will use the graphs of exponential functions, and apply the rules of exponents to solve exponential equations.

Activity 1: Maintaining Previous Skills with Exponential Expressions

Exponents were originally introduced into mathematics as a shorthand device for repeated multiplication. Here is an example defining a **power**.

$$\text{base} \rightarrow \underbrace{a^{n \leftarrow \text{exponent}}}_{\text{power}} \text{ means } \underbrace{a \times a \times a \times \ldots \times a}_{n \text{ times}}$$

The expression a^n reads ***a* to the exponent *n*** or ***a* to the n^{th} power**.

To operate with powers, you need some rules to follow. Such operations led to the development of the following laws of exponents.

Product Law	$a^m \times a^n = a^{m+n}$
Quotient Law	$a^m \div a^n = a^{m-n}$
Power of a Power Law	$\left(a^m\right)^n = a^{mn}$
Power of a Product Law	$(ab)^m = a^m b^m$
Power of a Quotient Law	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m},\ b \neq 0$

Note: The first two laws can be used only when the base is the **same**. There are three key definitions related to powers:

- $a^0 = 1$, provided $a \neq 0$
- $a^{-n} = \frac{1}{a^n}$ or $\frac{1}{a^{-n}} = a^n$, if $a \neq 0$
- $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$ or $\sqrt[n]{a^m}$

These laws and definitions can be used to simplify or evaluate expressions. They are used in the following examples.

Use the laws of exponents and the definitions to simplify the expressions in each example. Express all powers with positive exponents.

Example 1

Simplify $\dfrac{3^4 \times 3^3}{(3^3)^2}$.

Solution

$$\frac{3^4 \times 3^3}{(3^3)^2} = \frac{3^{4+3}}{3^{3\times 2}}$$

$$= \frac{3^7}{3^6}$$

$$= 3^{7-6}$$

$$= 3^1 \text{ or } 3$$

Example 2

Simplify $\left(\dfrac{2x}{3}\right)^2 \times \left(\dfrac{3y}{2}\right)^4$.

Solution

$$\left(\frac{2x}{3}\right)^2 \times \left(\frac{3y}{2}\right)^4 = \frac{2^2 \times x^2}{3^2} \times \frac{3^4 \times y^4}{2^4}$$

$$= \frac{2^2 \times x^2 \times 3^4 \times y^4}{3^2 \times 2^4}$$

$$= \frac{3^{4-2} \times x^2 \times y^4}{2^{-2} \times 2^4}$$

$$= \frac{3^2 \times x^2 \times y^4}{2^2}$$

$$= \frac{9x^2y^4}{4}$$

Example 3

Simplify $9^{\frac{3}{2}} \times 27^{-\frac{2}{3}}$.

Solution

$$9^{\frac{3}{2}} \times 27^{-\frac{2}{3}} = \frac{9^{\frac{3}{2}}}{27^{\frac{2}{3}}}$$

$$= \frac{\left(\sqrt{9}\right)^3}{\left(\sqrt[3]{27}\right)^2}$$

$$= \frac{3^3}{3^2}$$

$$= 3^1 \text{ or } 3$$

Example 4

Simplify $\left(\frac{a^3b^2}{b^{-2}}\right)^4$.

Solution

$$\left(\frac{a^3b^2}{b^{-2}}\right)^4 = \frac{a^{12}\times b^8}{b^{-8}}$$
$$= a^{12}\times b^8 \times b^8$$
$$= a^{12}b^{16}$$

In order to simplify expressions, it is sometimes necessary to convert all numbers to the same base before the laws can be used. Look at the following examples.

Example 5

Convert all the numbers of $\frac{5^3\times 25^4}{625^{\frac{1}{2}}}$ as powers of the same base; then simplify.

Solution

$$\frac{5^3\times 25^4}{625^{\frac{1}{2}}} = \frac{5^3\times\left(5^2\right)^4}{\left(5^4\right)^{\frac{1}{2}}}$$
$$= \frac{5^3\times 5^8}{5^2}$$
$$= \frac{5^{11}}{5^2}$$
$$= 5^9$$

Example 6

Convert all the numbers of $\frac{9^{2x}\times 3^{6x+1}}{\left(27^{x+1}\right)^3}$ as powers of the same base; then simplify.

Solution

$$\frac{9^{2x}\times 3^{6x+1}}{\left(27^{x+1}\right)^3}=\frac{\left(3^2\right)^{2x}\times 3^{6x+1}}{\left[\left(3^3\right)^{x+1}\right]^3}$$

$$=\frac{3^{4x}\times 3^{6x+1}}{\left[3^{3x+3}\right]^3}$$

$$=\frac{3^{10x+1}}{3^{9x+9}}$$

$$=3^{(10x+1)-(9x+9)}$$

$$=3^{x-8}$$

Example 7

Convert all the numbers of $9^{\frac{3}{2}}\times 27^{-\frac{2}{3}}$ as powers of the same base; then simplify.

Solution

$$9^{\frac{3}{2}}\times 27^{-\frac{2}{3}}=\left(3^2\right)^{\frac{3}{2}}\times\left(3^3\right)^{-\frac{2}{3}}$$

$$=3^{\frac{6}{2}}\times 3^{-\frac{6}{3}}$$

$$=3^3\times 3^{-2}$$

$$=3^1 \text{ or } 3$$

Compare the solution in Example 7 to the solution in Example 3.

When simplifying exponential expressions, it is **not** necessary to show each step. However, it is necessary to convert expressions to the same base when applying the **Product Law** or the **Quotient Law**. For example, $a^b\times b^a$ cannot be simplified further.

1. Use the laws of exponents to simplify the following expressions. Express with positive exponents.

a. $\left(3^2\right)^2\div\left(3^3\times 3^2\right)$

b. $\frac{\left(x^2yz\right)^2}{\left(xyz^2\right)^3}\times\left(\frac{2}{y}\right)^0$

c. $\left(\frac{x}{y}\right)^2\left(\frac{y}{z}\right)^3\left(\frac{y}{x}\right)^2$

d. $\frac{\left(t^{x-3}\right)\left(t^{2x+5}\right)}{t^2}$

e. $3^{\frac{1}{7}x}\times 27^{\frac{2}{7}x}\div 9^{2x-1}$

f. $\sqrt{8}+\sqrt[3]{32}$

g. $\sqrt[6]{4^3}\div\sqrt[6]{8^2}$

h. $\frac{a^{10}+a^9}{a^2+a}$

2. Show that the expression on the left side is equivalent to the expression on the right side.

a. $\frac{\left(3^{n+1}\right)\left(3^{n+2}\right)(3)}{9^n}=81$

b. $\frac{5^{4n}+5^{3n}}{5^n+5^{2n}}=5^{2n}$

c. $16^n+16^n+16^n+16^n=4^{2n+1}$

Check your answers by turning to the Appendix.

In Activity 1 you applied the laws of exponents. You will use some of these laws later in this module to define and graph exponential functions.

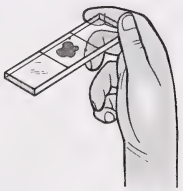
Activity 2: Identifying and Graphing Exponential Functions

The work with exponents will be expanded to include exponential functions. In an exponential function, the independent variable appears in the exponent of the power. For example, for $y = a^x$, a is the base and x is the independent variable.

In general, $y = a^x$ represents an **exponential function**, where $a > 0$ and $a \neq 1$.

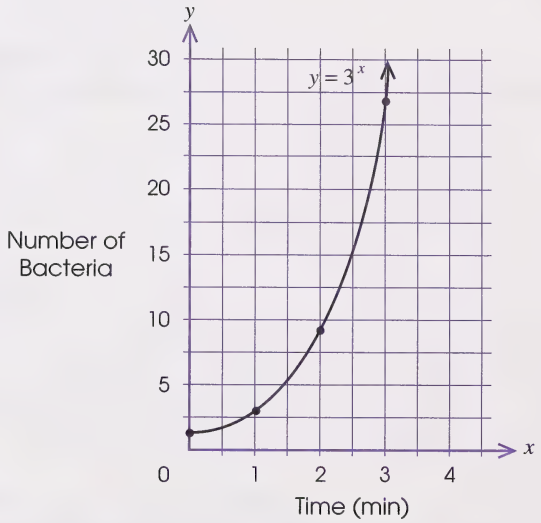
Now examine the following theory related to exponential growth.

A laboratory technician places a single bacterium on a slide. One minute later there are three bacteria. After two minutes, the bacteria have increased to nine. This example illustrates exponential growth. This growth function is represented by $y = 3^x$, where x is the number of minutes after the starting time.



The growth in the number of bacteria can be displayed on a graph where $y = 3^x$. Since the number of bacteria is **tripling** each minute, the base is 3. In the function $y = 3^x$, y represents the number of bacteria after x minutes. A table of values should be prepared before plotting the graph.

x	0	1	2	3
y	1	3	9	27



Why does the graph represent a function?

Recall the **vertical line test**. Any vertical line will intersect the graph of $y = 3^x$ only at one point. This means that the graph represents a function; that is, for each point on the graph, there is only one y-coordinate for each x-coordinate.

From a graph representing bacterial growth, it is possible to estimate the following:

- the number of bacteria after a given number of minutes
- the number of minutes to produce a given number of bacteria

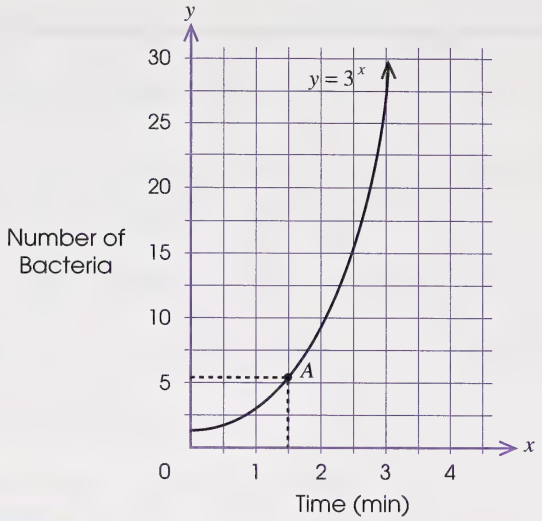
The following examples illustrate how these estimates are achieved.

Example 1

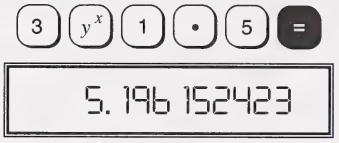
Refer to the graph $y = 3^x$. Point A represents the point on the curve when 1.5 min have elpased. Estimate the number of bacteria at that time.

Solution

Locate 1.5 on the x-axis. Draw a vertical line from 1.5 until it intersects the curve. This locates point A on this curve. From point A, draw a horizontal segment until it meets the y-axis. This segment intersects the y-axis at approximately 5.2.



You can check this approximation by using a calculator.



Thus, approximately 5 bacteria exist after 1.5 min.

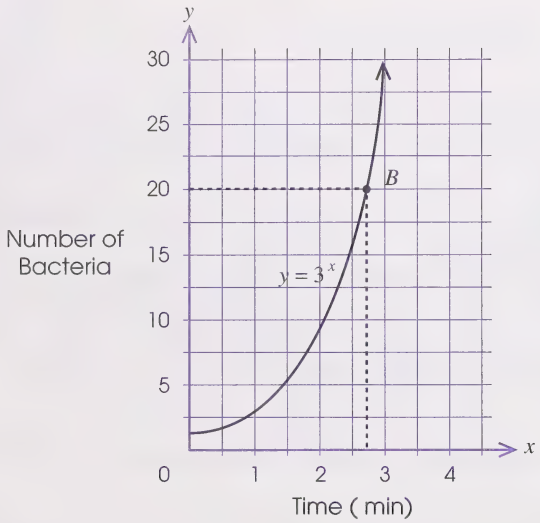
Note: Some calculators have the key x^y instead of y^x.

Example 2

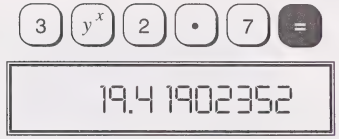
Point B represents the point on the curve when 20 bacteria have been produced. Estimate the time represented by point B (the time required to produce 20 bacteria).

Solution

Locate 20 on the y-axis. Draw a horizontal line from 20 until it intersects the graph. This locates point B on the graph. From point B, draw a vertical line until it meets the x-axis. This line segment intersects the x-axis at approximately 2.7.



Use a calculator to check this approximation.



After 2.7 min, there are about 20 bacteria.

You may use the graph $y = 3^x$ to solve other problems. See the solutions in the following examples.

Example 3

Use the graph of $y = 3^x$ to approximate $y = 3^{2.5}$; then use a calculator to check your approximations.

Solution

The procedure for the graphical solution of $y = 3^{2.5}$ is as follows:

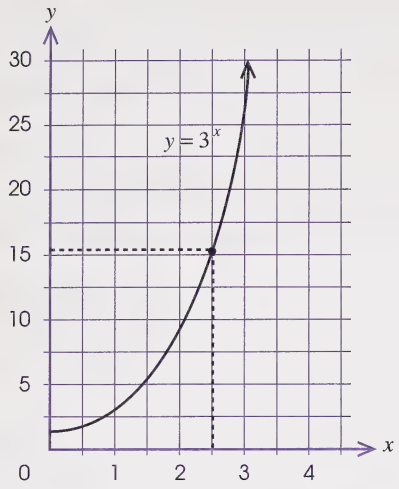
Step 1: Locate 2.5 on the x-axis.

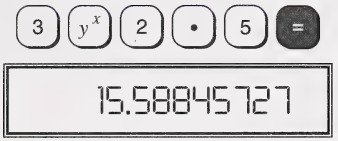
Step 2: Draw a vertical line to the curve.

Step 3: Draw a horizontal line from the point on the curve to the y-axis.

Step 4: Estimate the value on the y-axis.

From the graph, $y \doteq 16$.





Thus, y is approximately 16 when x is 2.5.

Example 4

Use the graph of $y = 3^x$ to find the approximate value of $12 = 3^x$; then use a calculator to verify your approximation.

Solution

The procedure for the graphical solution of $12 = 3^x$ is as follows:

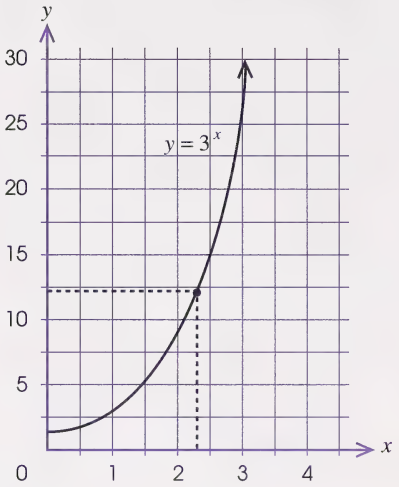
Step 1: Locate 12 on the y-axis.

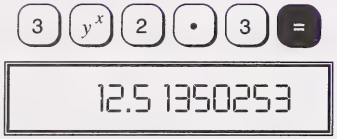
Step 2: Draw a horizontal line to the curve.

Step 3: Draw a vertical line from the point on the curve to the x-axis.

Step 4: Estimate the value on the x-axis.

From the graph, $x \doteq 2.3$.





Thus, x is approximately 2.3 when y is 12.

Example 5

How can the graph of $y = 3^x$ be used to estimate $y = 9^{1.4}$?

Solution

You must use the Power of a Power Law.

$$y = 9^{1.4}$$
$$= \left(3^2\right)^{1.4}$$
$$= 3^{2.8}$$

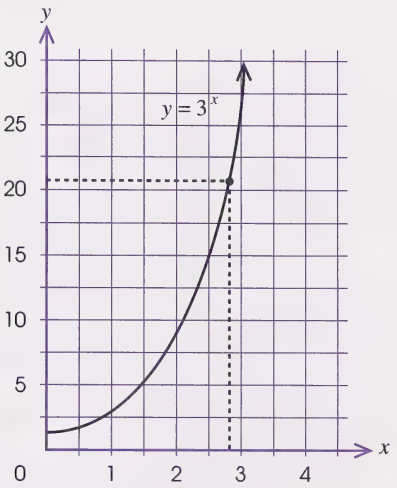
Step 1: Locate 2.8 on the x-axis.

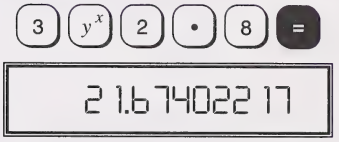
Step 2: Draw a vertical line to the curve.

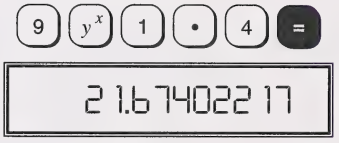
Step 3: Draw a horizontal line from the point on the curve to the y-axis.

Step 4: Estimate the value on the y-axis.

From the graph, $y \doteq 22$.







Thus, y is approximately 22 when the graphical method is used.

Use the graph $y = 3^x$ and the laws of exponents when solving the following problems. Express numbers as a power of 3 by using the graph or by using the laws of exponents. Then use the laws of exponents to simplify the expression. Finally, use the graph to approximate the final result.

Example 6

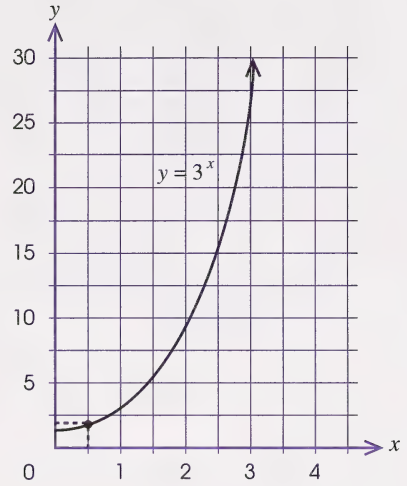
How can the graph of $y = 3^x$ be used to estimate $y = \sqrt{3}$?

Solution

$y = \sqrt{3}$

$= 3^{\frac{1}{2}}$ or $3^{0.5}$

From the graph, $3^{0.5} \doteq 1.7$.



Example 7

How can the graph of $y = 3^x$ be used to solve $y = 15 \times 3^{-1.2}$?

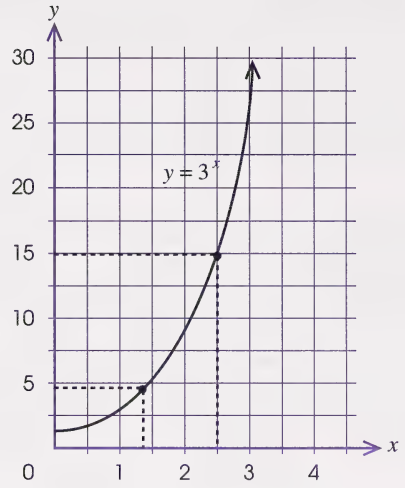
Solution

From the graph, $15 \doteq 3^{2.5}$.

$\therefore \; 15 \times 3^{-1.2} \doteq 3^{2.5} \times 3^{-1.2}$

$\doteq 3^{1.3}$

From the graph, $3^{1.3} \doteq 4$.



Example 8

How can the graph of $y = 3^x$ be used to estimate $y = \dfrac{20 \times 8}{4^3}$?

Solution

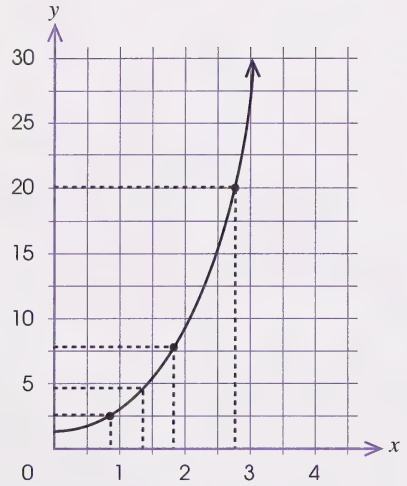
From the graph, $20 \doteq 3^{2.7}$, $8 \doteq 3^{1.9}$, and $4 \doteq 3^{1.3}$.

$$\therefore \frac{20 \times 8}{4^3} \doteq \frac{3^{2.7} \times 3^{1.9}}{\left(3^{1.3}\right)^3}$$

$$\doteq \frac{3^{4.6}}{3^{3.9}}$$

$$\doteq 3^{0.7}$$

$$\doteq 2.2$$



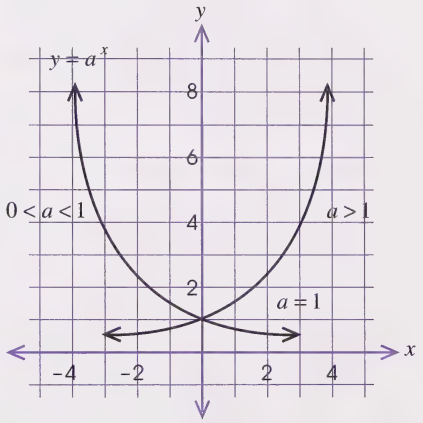
Note: The answer to Example 8 (2.2) is only an estimate. The actual answer is 2.5.

1. Graph the exponential functions $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$ on the same grid. Prepare a table of values before plotting the curves.

2. Use the graphs from question 1 to answer the following:

- **a.** State the x-intercepts for each function.
- **b.** State the y-intercepts for each function.
- **c.** State the domain of each function.
- **d.** State the range of each function.
- **e.** State the intersection point of the graphs of the functions.
- **f.** Do all exponential functions go through this point? Explain.

3. The general form of an exponential function can be defined as $y = a^x$, where $a > 0$ and $a \neq 1$. Using the graph of $y = a^x$, state whether each of the following statements is True or False.



- **a.** $(0,1)$ is common to all exponential functions.
- **b.** If $a > 1$, then y decreases when x increases.

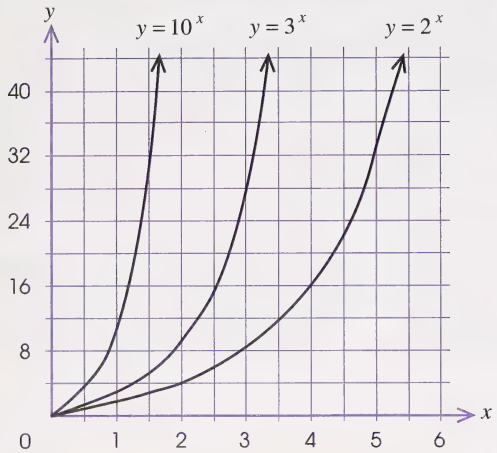
c. If $a = 1$, the function is linear (e.g., the graph of the function is a straight line). This is why $y = 1^x$ is not considered an exponential function.

d. If $a < 1$, the y-value is decreasing when x increases.

e. $(1, a)$ is a point on every graph of an exponential function.

f. $(2, a^2)$ is a point on every graph of an exponential function.

4. Use the graphs of three exponential functions to answer the following questions. Estimate the value of the variable. Use a calculator to check your approximation.



a. $2^x = 15$

b. $10^{1.5} = y$

c. $3^x = 40$

d. $9^x = 30$ (Hint: Convert to base 3 first.)

e. $8^{0.8} = y$ (Hint: Convert to base 2 first.)

Note: Because these are approximations, your answers may differ from the approximations in the solutions.

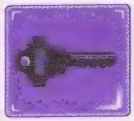
Check your answers by turning to the Appendix.

Now you know how to use the exponential graphs to estimate a given value.

Activity 3: Solving Exponential Equations

In this activity you will extend the concept of exponential functions and the laws of exponents to help you solve **exponential equations**. You will limit your discussion to equations where all powers can be expressed in terms of the **same** base.

The solution of exponential equations is based on the following property of powers and their exponents.



If $x^m = x^n$ and $x \neq 0$ or 1, then $m = n$.

Example 1

Solve $3^x = \left(\frac{1}{27}\right)^{\frac{3}{2}}$ for x. Use a calculator to check your answer.

Solution

$3^x = \left(\frac{1}{27}\right)^{\frac{3}{2}}$

$3^x = \left(27^{-1}\right)^{\frac{3}{2}}$ $\left(\frac{1}{a^n} = a^{-n}\right)$

$3^x = 27^{-\frac{3}{2}}$ (Power of a Power Law)

$3^x = \left(3^3\right)^{-\frac{3}{2}}$ (27 is expressed as a power of 3.)

$3^x = 3^{-\frac{9}{2}}$ (Power of a Power Law)

If $x^m = x^n$, then $m = n$. Therefore, $x = -\frac{9}{2}$ or -4.5.

Example 2

Solve $4^{2x} = 2^{x(x-2)}$ for x.

Solution

$4^{2x} = 2^{x(x-2)}$

$\left(2^2\right)^{2x} = 2^{x^2-2x}$ (4 is expressed as a power of 2.)

$2^{4x} = 2^{x^2-2x}$ (Power of a Power Law)

$\therefore\ 4x = x^2 - 2x$ $\left(\text{If } x^m = x^n, \text{ then } m = n.\right)$

$0 = x^2 - 6x$ $(-4x$ is added to both sides.)

$0 = x(x-6)$ (Factor the right side.)

If $0 = a \times b$, then $a = 0$ or $b = 0$. Therefore, $x = 0$ and $x = 6$.

Can you have two solutions? Check to see if there are two solutions.

If $x = 0$,

LS	RS
$4^{2(0)}$	$\left(2^0\right)^{0-2}$
$= 4^0$	$= 2^0$
$= 1$	$= 1$

LS = RS

If $x = 6$,

LS	RS
$4^{2(6)}$	$\left(2^6\right)^{6-2}$
$= 4^{12}$	$= 2^{24}$
$= 16\,777\,216$	$= 16\,777\,216$

LS = RS

Yes, there can be two solutions!

Example 3

Solve $125\left(25^{1-x}\right)=1$ for x.

Solution

$125\left(25^{1-x}\right)=1$

$5^3\left(5^2\right)^{1-x}=1$ (Express 125 and 25 as powers of 5.)

$5^3\times5^{2-2x}=1$ (Power of a Power Law)

$5^{3+2-2x}=1$ (Product Law)

$5^{5-2x}=5^0$ $\left(5^0=1,\ \text{since } a^0=1.\right)$

$\therefore\ 5-2x=0$ $\left(\text{If } x^m=x^n,\ \text{then } m=n.\right)$

$5=2x$

$x=\frac{5}{2}$ or 2.5

For any real number a, $a\neq0$, $a^0=1$.

You may be asking what happens if the bases cannot be made to be the same. This question will be answered later in this unit.

1. Solve the following exponential equations for x.

a. $7^{x+1}=343$

b. $3^{x+2}=81$

c. $2^{3x}=\frac{1}{2}$

d. $5^{2x}=\frac{1}{5}$

e. $4^{2x}=\left(8^x\right)^{x-2}$

f. $9^{3x}=\left(27^x\right)^{x-3}$

2. Solve the following exponential equations for x.

a. $2\left(5^{2x-9}\right)=250$

b. $3\left(7^{2x-5}\right)=147$

c. $3^{x(x-4)}=243$

d. $5^{x(x+2)}=125$

e. $\left(\frac{1}{9}\right)^{1.5x}=\left(\frac{1}{27^x}\right)^{x-1}$

f. $\left(\frac{1}{16}\right)^{3x}=\left(\frac{1}{64^x}\right)^{x-2}$

Check your answers by turning to the Appendix.

In this activity, the importance of the laws of exponents for solving exponential equations was emphasized. In the next activity you will study how to write and solve an exponential function given a real-world situation.

Activity 4: Writing and Solving Exponential Equations

Exponential functions are used to calculate compound interest, population growth, and economic growth. These functions are also used to predict the time for radioactive materials to decay. Another use is to describe the growth time of bacteria; for example, the **doubling period** of a certain bacteria. The doubling period is the period of time required for a given initial number of bacteria to double.

Example 1

A laboratory technician notes that a certain bacteria has a doubling period of 4 h. If there is an initial amount of 500 bacteria, how many will there be in 24 h?

Solution

Let A_0 be the initial number of bacteria.

In 4 h the number of bacteria becomes $2A_0$.

In 8 h the number of bacteria becomes $2 \times 2A_0 = 2^2 A_0$.

In 12 h the number of bacteria becomes $2 \times 2^2 A_0 = 2^3 A_0$.

In 16 h the number of bacteria becomes $2 \times 2^3 A_0 = 2^4 A_0$.

In 20 h the number of bacteria becomes $2 \times 2^4 A_0 = 2^5 A_0$.

In 24 h the number of bacteria becomes $2 \times 2^5 A_0 = 2^6 A_0$.

A_0 is the initial number of bacteria, A is the number after a time t, and p is the period of time for the doubling to occur. Both t and p must have the same units.

$$A = A_0 \, 2^{\frac{t}{p}}$$

Note that the 2 in the formula results from the doubling time.

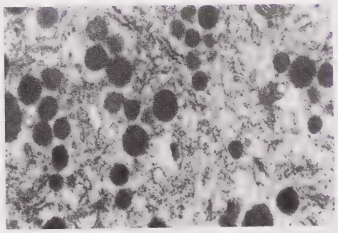
For example, $A_0 = 500$, $t = 24$ h, and $p = 4$ h.

$$\begin{aligned} A &= 500(2)^{\frac{24}{4}} \\ &= 500(2)^6 \\ &= 500(64) \\ &= 32\,000 \end{aligned}$$

Thus, 32 000 bacteria will exist in 24 h.

Example 2

A certain kind of yeast will grow to twenty-seven times its initial amount in 30 min. Find the time required for the yeast to triple.



Solution

$A = A_0\left(3^{\frac{t}{p}}\right)$

Note that 3 is used in the equation since the tripling period is required.

Let $A_0 = 1$; thus, $A = 27$ and $t = 30$.

$$27 = 1\left(3^{\frac{30}{p}}\right)$$

$$3^3 = 3^{\frac{30}{p}}$$

$$\therefore\ 3 = \frac{30}{p}$$

$$p = 10$$

Exponents are equal when the bases are the same.

The tripling period is 10 min.

Examples 3 and 4 concern **half-life**. A half-life is the period of time during which a given amount of a radioactive element decays until one-half of it is left.

Example 3

A radioactive element has a half-life of 8 d. Find the amount of this radioactive element remaining from an original amount of 55 g after 40 d.

Solution

In 8 d, the amount remaining is $55 \times \frac{1}{2} = 55 \times 2^{-1}$.

In 16 d, the amount remaining is $\left(55 \times 2^{-1}\right) \times \frac{1}{2} = 55 \times 2^{-2}$.

In 24 d, the amount remaining is $\left(55 \times 2^{-2}\right) \times \frac{1}{2} = 55 \times 2^{-3}$.

In 32 d, the amount remaining is $\left(55 \times 2^{-3}\right) \times \frac{1}{2} = 55 \times 2^{-4}$.

In 40 d, the amount remaining is $\left(55 \times 2^{-4}\right) \times \frac{1}{2} = 55 \times 2^{-5}$.

Let a_0 be the original amount of radioactive element, a be the amount remaining after a time t, and h be the half-life period in the same units as t. The amount of radioactive element remaining can be expressed as $a = a_0\left(2^{-\frac{t}{h}}\right)$.

For this example, $a_0 = 55$ g, $t = 40$ d, and $h = 8$ d.

$$a = 55\left(2^{-\frac{40}{8}}\right)$$

$$= 55\left(2^{-5}\right)$$

$$= 55 \times \frac{1}{2^5}$$

$$= \frac{55}{32}$$

$$= 1\frac{23}{32}\ \text{g}$$

The amount remaining after forty days is $1\frac{23}{32}$ g.

Example 4

In four days an initial amount of radioactive substance decays to $\frac{1}{2\sqrt{2}}$ of its initial amount. Find the half-life of this radioactive substance.

Solution

$a = a_0\left(2^{-\frac{t}{h}}\right)$ and $a = \frac{1}{2\sqrt{2}}a_0$ ($t = 4$ and h must be determined.)

$$\frac{1}{2\sqrt{2}}a_0 = a_0\left(2^{-\frac{4}{h}}\right)$$

$$\frac{1}{2\sqrt{2}} = 2^{-\frac{4}{h}}$$

$$2^{-\frac{3}{2}} = 2^{-\frac{4}{h}}$$

$$\frac{1}{2^1 \times 2^{\frac{1}{2}}} = \frac{1}{2^{\frac{3}{2}}} = 2^{-\frac{3}{2}}$$

$\therefore -\frac{3}{2} = -\frac{4}{h}$

The exponents are equal since the bases are the same.

$3h = 8$

$h = 2\frac{2}{3}$

The half-life of this radioactive substance is $2\frac{2}{3}$ d.

1. A laboratory sample initially contains 120 bacteria. After 3 h the sample grows to 1920 bacteria. Find the doubling period for this sample.

2. A bacteria sample is allowed to grow; and after 8 h, there are approximately 1620 bacteria. If the amount of bacteria triples every 2 h, find the following:

a. the initial amount of bacteria
b. the number of bacteria after 16 h

3. A radioactive substance has a half-life of 24 min.

a. What amount of radioactive substance remains after 8 min?
b. What is the amount remaining after 72 min?

4. A radioactive substance has a half-life of 4.6 d. Find the time required for 4 g of the substance to deteriorate to 1 g.

Check your answers by turning to the Appendix.

The exponential rules not only apply to the exponential equations, but also to exponential functions. In some cases a calculator may speed up your calculations.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

You must become familiar with the laws of exponents to simplify exponential expressions and solve some types of exponential equations. Some of these laws will be useful in your study of the sections that follow in this module.

Test your memory by completing the following exercises.

For questions 1 and 2, fill in the blanks.

1. **a.** The _____ Law is represented by $a^m \cdot a^n =$_____.

b. The _____ Law is represented by $\left(a^m\right)^n =$_____.

c. The _____ Law is represented by $(ab)^m =$_____.

d. The _____ Law is represented by $a^m \div a^n =$_____.

e. The _____ Law is represented by $\left(\frac{a}{b}\right)^m =$_____.

2. **a.** $a^0 =$_____

b. $a^{-n} =$_____ or $\frac{1}{a^{-n}} =$_____

c. $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$ or _____

How many of the laws did you remember? Did you notice something special about growth and decay problems? Do the following exercises to answer this question.

3. Draw the graph of $y = 2^x$ to represent the growth curve. Describe the graph with respect to y and x values.

4. Draw the graph of $y = \left(\frac{1}{2}\right)^x$ which is the graph of a decay curve. What happens to the curve when x increases?

5. Complete the following sentences so that you may have a firmer grasp of the growth and decay problems.

a. The doubling period is the period of _____ required for a given quantity to double in size.

b. Half-life is the period of time during which a given amount of radioactive material decays until _____ of it is left.

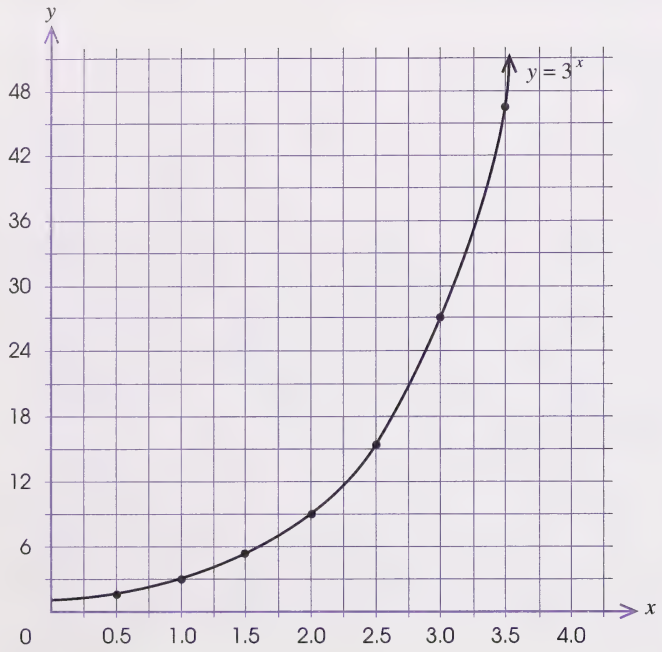
6. The amount of bacteria triples in size every 4 h. The initial number of bacteria is 500. Determine the number of bacteria after 16 h.

7. Determine the amount of radioactive substance remaining from an original amount of 4 g after 60 d. The half-life period is 10 d.

Check your answers by turning to the Appendix.

Enrichment

Data from exponential graphs can be used with laws of exponents to find approximate values in computations. Refer to Examples 6, 7, and 8 in Activity 2 of this section. They illustrate simple computations using the combined operations. This combined method can be expanded to find powers and roots of numbers.



Example 1

Use the graph of $y = 3^x$ and the laws of exponents to find the approximate value of each of the following:

- $4.5 \times 3^{1.75}$
- $34^{\frac{1}{4}}$

Solution

From the graph, $4.5 \doteq 3^{1.37}$.

$$\therefore \ 4.5 \times 3^{1.75} = 3^{1.37} \times 3^{1.75}$$
$$= 3^{1.37+1.75}$$
$$= 3^{3.12}$$

Using the graph, $3^{3.12} \doteq 31$.

$34^{\frac{1}{4}} \doteq \left(3^{3.2}\right)^{\frac{1}{4}}$ (Use the graph to change 34 to a power of 3.)

$\doteq 3^{0.8}$ (Power of a Power Law)

$\doteq 2.5$ (Locate 0.8 on the x-axis and obtain 2.5 on the y-axis.)

Example 2

Solve $3^x = 141$.

Solution

The graph of $y = 3^x$ does not have y-values greater than 48. In order to use this graph, express 141 as the product of a power of 3 and a number less than 50.

$3^x = 141$ $\quad (141 \div 27 \doteq 5.2)$

$3^x \doteq 27 \times 5.2$ (Use the graph to express 5.2 as a power of 3.)

$3^x \doteq 3^3 \times 3^{1.5}$ (Product Law)

$3^x \doteq 3^{4.5}$ (Exponents are equal when bases are equal.)

$x \doteq 4.5$

1. Use the graph of $y = 3^x$ and the laws of exponents to find the approximate values of each of the following:

a. $18^{\frac{1}{5}}$ b. $\sqrt{6^3}$ c. $(10)^{\frac{1}{3}} \times \sqrt[3]{14^2}$

2. Use the graph of $y = 3^x$ and the laws of exponents to solve $3^{x+1} = 73$.

Check your answers by turning to the Appendix.

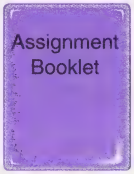
Conclusion

In this section, the exponential function was investigated. You were shown that changing the base of an exponential function affects its graph. The graph of an exponential function was used to solve simple exponential equations. In addition, you reviewed the laws of exponents, which had been studied in previous courses. Finally, you applied your knowledge of exponential functions to real-world problems such as radioactive decay and exponential growth.

Laboratory technicians must understand the principles from this section. When they look through their respective microscope at a slide prepared from a bacterial culture, shape and size aid them in identifying the specimen. Lab technicians describe the size of a bacteria by using exponential notation. They also know that the growth of a bacteria in a culture may be modelled using an exponential function.

In the next section you will discover that logarithmic functions can be defined in terms of exponential functions, and that logarithms will help you solve problems in exponential growth and decay.

Assignment



You are now ready to complete the section assignment.

Section 2: Introduction to Logarithms

THE BETTMAN ARCHIVE

The eccentric Scottish mathematician **John Napier** (1550–1617) is credited for developing logarithms—a system which permitted complex arithmetic calculations (involving products, quotients, powers, and roots) to be performed simply and quickly through the use of tables. **Henry Briggs** (1561–1630), who collaborated with Napier, suggested that logarithms be based on powers of 10.

Jobst Bürgi (1551–1632), a Swiss mathematician working independently of Napier and Briggs, is also credited for developing logarithms. He drafted logarithm tables and published his results in 1620.

The need for logarithms arose in the early seventeenth century to solve problems in astronomy, navigation, and compound interest. In this section you will define and graph logarithmic functions, explore relationships between exponential and logarithmic functions, and solve equations using logarithms.

Activity 1: Defining Logarithmic Functions

Exponential equations occur so frequently in applications that mathematicians have developed a quick method to solve them. This method involves logarithms. You are familiar with exponents; the definition of a logarithm will be based on the inverse of the exponential function. The inverse of an exponential function, such as $y = 3^x$, is a reflection about the line $y = x$. This gives you another function, $x = 3^y$. These two functions are shown in Example 1.

Example 1

Sketch the graphs of $y = 3^x$ and $x = 3^y$.

Solution

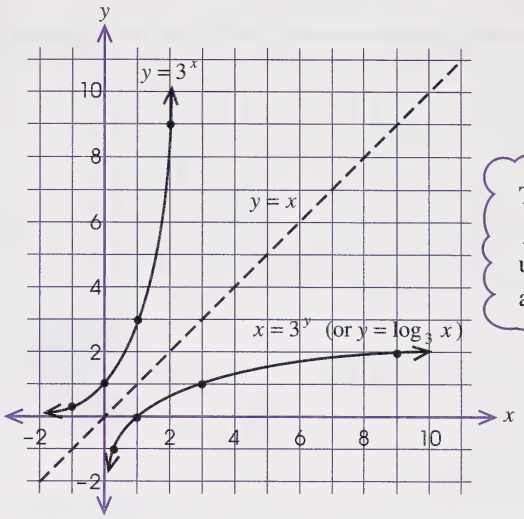
The sketch of $y = 3^x$ can be obtained by choosing appropriate values for x and solving for y (that is, the table-of-values method). In order to plot the curve for $x = 3^y$, choose values for y and solve for x.

$y = 3^x$

x	-1	0	1	2
y	$\frac{1}{3}$	1	3	9

$x = 3^y$

x	$\frac{1}{3}$	1	3	9
y	-1	0	1	2



The equation $x = b^y$ is usually written as $y = \log_b x$.

Use Zap-a-Graph™ to draw the preceding graphs. How do they compare?

Example 2

Use the graphs in Example 1 to answer the following:

- What happens to y in each function as x increases?
- State the domain and range for $y = 3^x$.
- State the domain and range for $x = 3^y$.
- Along what line could a mirror be placed so that one curve reflects into the other?

Solution

For both functions, the value of y increases as x increases.

For $y = 3^x$, the domain is $\{x | x \in R\}$.
For $y = 3^x$, the range is $\{y | y > 0,\ y \in R\}$.

For $x = 3^y$, the domain is $\{x | x > 0,\ x \in R\}$.
For $x = 3^y$, the range is $\{y | y \in R\}$.

Remember: The domain is the set of all first elements of the ordered pairs of the points on the curve. The range is the set of all second elements of the ordered pairs on the curve.

The equation of the line is $y = x$.

Note that the range of $y = 3^x$ is the domain of $x = 3^y$, and the domain of $y = 3^x$ is the range of $x = 3^y$.

The inverse of the exponential function $y = b^x$, for $b > 0$ and $b \neq 1$, can be obtained algebraically by interchanging x and y to give $x = b^y$.

When $x = b^y$ is written with y as the subject, the function is $y = \log_b x$. This is read as ***y* is the logarithm of *x* to the base *b***.

Definition of Logarithmic Function

$x = b^y \Leftrightarrow y = \log_b x$, where $b > 0$ and $b \neq 1$

The symbol $\Leftrightarrow$ reads **if and only if**.

Note: The value of the logarithm is the exponent of the corresponding exponential function.

1. Write the inverse of the following exponential functions.

a. $y = 5^x$ **b.** $x = 3^y$ **c.** $q = r^s$ **d.** $c = d^g$

2. Given the exponential function $y = a^d$, do the following:

a. State the inverse of $y = a^d$.
b. Change the inverse from exponential to logarithmic form.

Check your answers by turning to the Appendix.

You saw how a logarithm can be defined using the exponential function idea. In the next activity you will graph logarithmic functions.

Activity 2: Identifying and Graphing Logarithmic Functions

The graph in Example 1 shows you that the logarithmic function is related to the exponential function.

Example 1

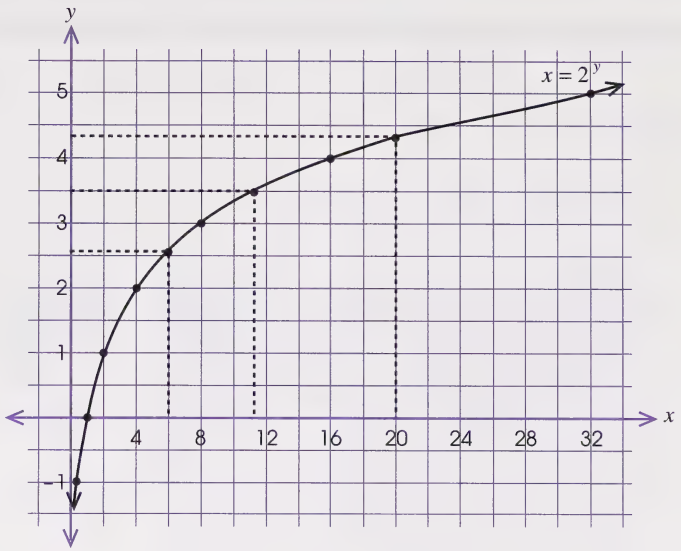
Draw the graph of $y = \log_2 x$.

Solution

Change $y = \log_2 x$ to exponential form, and choose values for y to get corresponding x-values.

$x = 2^y$

x	$\frac{1}{2}$	1	2	4	8	16	32
y	-1	0	1	2	3	4	5



Example 2

Using the graph in Example 1, approximate the value of $\log_2 6$.

Solution

Locate 6 on the x-axis. Draw a vertical line from 6 until it intersects the curve. From this point on the curve, draw a horizontal line until it meets the y-axis. An approximate value for $\log_2 6$ is 2.6.

Broken lines on the graph show where the y-value is located.

$\therefore\ 2^{2.6} \doteq 6$

Example 3

Use the graph in Example 1 to approximate the value of $\log_2 20$.

Solution

Locate 20 on the x-axis. Draw a vertical line from 20 until it intersects the curve. Then draw a horizontal line until it intersects the y-axis. Thus, $\log_2 20 \doteq 4.3$.

$\therefore\ 20 \doteq 2^{4.3}$

Example 4

Use the graph in Example 1 to approximate the value of x when $y = 3.5$ in $x = 2^{3.5}$.

Solution

Locate 3.5 on the y-axis. Draw a horizontal line from 3.5 until it intersects the curve. Then draw a vertical line from this point on the curve until it intersects the x-axis. The approximate value of x is 11. Thus, $11 \doteq 2^{3.5}$.

1. Find the domain and range for the function $y = \log_b x$.

2. If $P\left(a, b^a\right)$ is a point on the graph of $y = b^x$, find the coordinates of a corresponding point Q on the graph of $y = \log_b x$.

3. For each statement, state whether it is True or False. Correct each false statement.

a. $y = \log_2 x$ is the logarithmic function with base 2.

b. $(1, 0)$ is a point common to all logarithmic functions.

c. $y = \log_b x$ is equivalent to $b^y = x$.

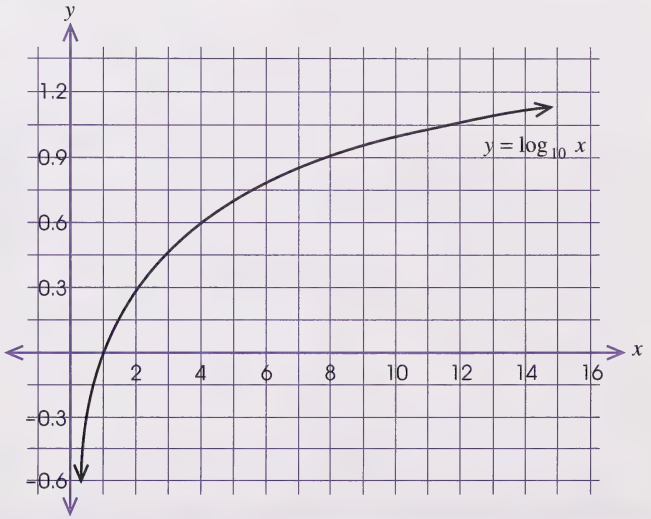
d. If $x = 0$, then $\log_b x = 0$.

e. $y = \log_b x$ is not defined for $x < 0$.

f. The inverse of $\log_5 x = y$ could be written as $\log_5 y = x$.

4. Draw the graphs of $y = \left(\frac{1}{2}\right)^x$ and its inverse on the same axes.

5. Use the graph of $y = \log_{10} x$ to answer the following questions.



a. Find the approximate value for $\log_{10} 4$.

b. Find the approximate value for $\log_{10} 8$.

c. Find the approximate value for $\log_{10} 10$.

d. If y is 0.15, find the approximate value of x that satisfies the equation $10^{y} = x$.

e. If y is 0.75, find the approximate value of x that satisfies the equation $10^{y} = x$.

Use a calculator to find the value for $10^{0.15}$ and $10^{0.75}$. Compare your calculated values to the approximate values obtained by using the graph.

Check your answers by turning to the Appendix.

Did you notice that $x > 0$ for $\log_{10} x$? The calculator is a useful tool to check the approximate values obtained from the graph.

Activity 3: Transforming an Exponential Form into Logarithmic Form and Vice Versa

The definition of the logarithmic function is used to express the exponential equations in logarithmic form and vice versa.

GENERAL RULE	
Exponential Form	**Logarithmic Form**
$x = a^{y}$ Base ***a*** is raised to the exponent ***y*** and the answer is ***x***.	$y = \log_{a} x$ ***y*** is the exponent to which you raise base ***a*** to get the answer ***x***.

At this point, the puzzle may be a little confusing. Take a moment to examine it.

The inverse of $y = a^{x}$ is $x = a^{y}$.

This inverse in exponential form can be written in the logarithmic form $y = \log_{a} x$.

$x = a^{y}$ means $y = \log_{a} x$

$\therefore\ c = a^{b} \Leftrightarrow b = \log_{a} c$

$y = a^x$ and $y = \log_a x$ are inverses.

$y = \log_a x$ and $x = a^y$ are equivalent.

You can change a logarithmic form to an exponential form and vice versa. This is shown in the following examples.

Example 1

Express $x = 3^y$ in logarithmic form.

Solution

$x = 3^y$

Therefore, $\log_3 x = y$.

$x = a^y \Leftrightarrow y = \log_a x$

Example 2

Express $125 = 5^3$ in logarithmic form.

Solution

$125 = 5^3$

Therefore, $\log_5 125 = 3$.

Example 3

Express $3^{-2} = \frac{1}{9}$ in logarithmic form.

Solution

$3^{-2} = \frac{1}{9}$

Therefore, $\log_3 \frac{1}{9} = -2$.

Example 4

Express $\log_a m = t$ in exponential form.

Solution

$\log_a m = t$

Therefore, $m = a^t$.

Example 5

Express $\log_2 32 = 5$ in exponential form.

Solution

$\log_2 32 = 5$

Therefore, $32 = 2^5$.

Example 6

Express $\log_9 \frac{1}{81} = -2$ in exponential form.

Solution

$\log_9 \frac{1}{81} = -2$

Therefore, $\frac{1}{81} = 9^{-2}$.

The statement $\log_{10} 100 = 2$ is an example of a special logarithmic statement that you can call the **common logarithm**. This form is useful in solving problems found later in this unit. The statement $\log_{10} 100 = 2$ may be written as $\log 100 = 2$. The base 10 is understood although it is not indicated. Thus, the general form of the function for the common logarithm is $y = \log x$.

In $y = \log_{10} x$, the domain of x is all positive numbers and y, the range of the logarithm, consists of all real numbers. Thus, 0 and negative numbers do not have logarithms.

1. Write the following in logarithmic form.

a. $3^4 = 81$ **b.** $16 = 2^4$ **c.** $32 = 2^5$

d. $4^3 = 64$ **e.** $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$ **f.** $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$

g. $2^{-3} = \frac{1}{8}$ **h.** $3^{-2} = \frac{1}{9}$ **i.** $4^0 = 1$

j. $1 = 6^0$ **k.** $a^b = c$ **l.** $d = b^a$

2. Write the following in exponential form.

a. $\log_2 16 = 4$ **b.** $\log_3 81 = 4$

c. $\log_4 64 = 3$ **d.** $\log_5 25 = 2$

e. $\log_5 \frac{1}{5} = -1$ **f.** $\log_4 \frac{1}{4} = -1$

g. $\log_{10} 100 = 2$ **h.** $\log_{10} 10\,000 = 4$

i. $\log_a b = c$ **j.** $\log_c d = e$

k. $\log_{10} 0.01 = -2$ **l.** $\log_{10} 0.1 = -1$

Check your answers by turning to the Appendix.

Changing exponential equations to logarithmic functions, and vice versa, are useful in solving logarithmic equations.

Activity 4: Solving Logarithmic Equations Using Exponential Form

To solve a logarithmic equation, convert it to an exponential form and solve for the unknown variable. For instance, the equation $\log_7 x = 4$ is changed to the exponential form.

$$\log_{\underset{\text{base}}{7}} x = \overset{\text{exponent}}{4} \Leftrightarrow x = 7^{\underset{\text{value}}{4}}$$

You are now going to look at a series of examples which illustrate different types of questions involving logarithms. Note the method of converting between exponential and logarithmic forms when solving problems dealing with logarithms.

Example 1

Solve for x when $\log_3 81 = x$.

Solution

$$\log_3 81 = x$$
$$3^x = 81$$
$$3^x = 3^4$$
$$x = 4$$

Remember: When bases are equal, the exponents are equal, and vice versa.

Example 2

Solve for x when $\log_2 64 = x$.

Solution

$$\log_2 64 = x$$
$$2^x = 64$$
$$2^x = 2^6$$
$$x = 6$$

Example 3

Evaluate $\log_{27} 3$.

Solution

Let $x = \log_{27} 3$.

$\therefore\ 27^x = 3$

$(3^3)^x = 3$

$3^{3x} = 3^1$

$3x = 1$

$x = \frac{1}{3}$

Therefore, $\log_{27} 3 = \frac{1}{3}$.

Example 4

Solve for x when $x = \log_4 \sqrt[3]{8}$.

Solution

$x = \log_4 \sqrt[3]{8}$ means 4 raised to what exponent gives $\sqrt[3]{8}$.

$4^x = \sqrt[3]{8}$

$4^x = 2$

$(2^2)^x = 2$

$2^{2x} = 2^1$

$2x = 1$

$x = \frac{1}{2}$

Example 5

Solve for x when $\log_{\sqrt{5}} 125 = x$.

Solution

$\log_{\sqrt{5}} 125 = x$

$(\sqrt{5})^x = 125$

$(\sqrt{5})^x = 5^3$

$(\sqrt{5})^x = \left[(\sqrt{5})^2\right]^3$

$(\sqrt{5})^x = (\sqrt{5})^6$

$x = 6$

Change 5 to a common base.
$5 = (\sqrt{5})^2$

Therefore, $\log_{\sqrt{5}} 125 = 6$.

1. Solve the following.

a. $x = \log_9 3$

b. $\log_8 16 = y$

c. $\log_{10} 0.001 = m$

d. $x = \log_{\sqrt{3}} 27$

e. $y = \log_{10} 0.1$

f. $y = \log_{\sqrt{2}} 16$

g. $x = \log_9 \sqrt[4]{81}$

h. $m = \log_{125} \sqrt[3]{25}$

Check your answers by turning to the Appendix.

Take a closer look at Example 1. Did you know that $\log_3 81$ can be written in the form $\log_3 3^4$?

What does the exponent 4 in 3^4 tell you? The solution of $\log_3 81$ is the same as the exponent 4 in 3^4.

Now examine Example 2.

$$\log_2 64 = \log_2 2^6$$

The solution of $\log_2 = 64$ is the same as the exponent 6 in 2^6.

You come to the following conclusion.

When solving equations of the form $x = \log_a a^m$, x has the value of the exponent m.

Use this definition to solve the next two equations in the following examples.

Example 6

Evaluate $x = \log_5 5^{37}$.

Solution

From the definition, $x = 37$.

Example 7

Evaluate $x = \log_{10} 1000$.

Solution

$$x = \log_{10} 1000$$
$$= \log_{10} 10^3$$
$$= 3$$

2. Evaluate the following.

a. $\log_4 16$ **b.** $\log_3 729$

c. $\log_7 49$ **d.** $\log_2 \frac{1}{8}$

Check your answers by turning to the Appendix.

There is yet another type of logarithm that can be evaluated very quickly. You will study this in the next example.

Example 8

Evaluate $x = 3^{\log_3 81}$.

Solution

$\log_3 81 = 4$ since $3^4 = 81$.

$\therefore\ x = 3^{\log_3 81}$

$= 3^4$

$= 81$

This leads to the following rule.

When solving equations of the form $x = a^{\log_a d}$, then x has the value of d.

3. Evaluate the following questions. You may use a calculator to verify some of the results.

a. $8^{\log_5 25}$ **b.** $2^{\log_2 32}$

c. $\log_4 (\log_4 16)$ **d.** $\log_{\frac{1}{3}} (\log_4 4^9)$

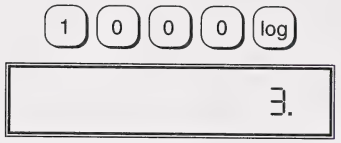
Check your answers by turning to the Appendix.

Use a calculator to answer Examples 9, 10, and 11.

Example 9

Evaluate $\log_{10} 1000$.

Solution

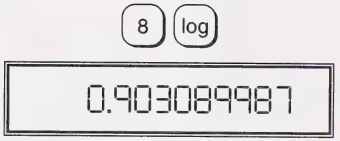


Remember: The log key on a calculator refers to base 10 logarithms only. This means $10^3 = 1000$.

Example 10

Evaluate $\log 8$ to two decimal places.

Solution



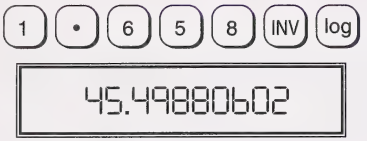
$\therefore\ 8 = 10^{0.903\,089\,987}$

The result is $\log 8 = 0.90$ to two decimal places.

Example 11

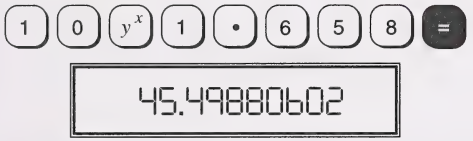
Evaluate $10^{1.658}$ to two decimal places.

Solution



The result is 45.50 to two decimal places.

You can also solve Example 11 by doing the following:



Use a calculator to answer question 4.

4. Evalute the following questions to two decimal places.

a. $\log_{10} 15$ **b.** $\log_{10} 20$

c. $10^{4.27}$ **d.** $10^{3.69}$

e. $\log_{10} 124.5 + \log_{10} 35.4$ **f.** $\log_{10} 147.8 + \log_{10} 62.8$

g. $\dfrac{\log_{10} 12.83}{\log_{10} 27.45}$ **h.** $\dfrac{\log_{10} 15.08}{\log_{10} 46.35}$

i. $10^{0.85}$ **j.** $10^{0.37}$

k. $10^{1.57}$ **l.** $10^{2.03}$

For questions 5 to 14, do either a or b. Solve for the unknown in each problem.

5. a. $\log_4 x = 3$ **b.** $\log_3 x = 2$

6. a. $\log_9 x = \dfrac{1}{2}$ **b.** $\log_{16} x = \dfrac{1}{2}$

7. a. $\log_x 8 = 3$ **b.** $\log_x 64 = 2$

8. a. $\log_5 5^{1.4} = x$ **b.** $\log_6 6^{2.8} = x$

9. a. $\log_x 125 = 3$ **b.** $\log_x 64 = 3$

10. a. $\log_9 x = -\dfrac{3}{2}$ **b.** $\log_{16} x = -\dfrac{3}{4}$

11. a. $\log_x \dfrac{1}{16} = -\dfrac{2}{3}$ **b.** $\log_x \dfrac{1}{32} = -\dfrac{5}{2}$

12. a. $x = \log_4 64^{-1}$ **b.** $x = \log_5 25^{-\frac{3}{5}}$

13. a. $x = \log_3 \sqrt[5]{3}$ **b.** $x = \log_5 25^{\frac{1}{5}}$

14. **a.** $x = 3^{\log_3 \frac{1}{9}}$ **b.** $x = 5^{\log_5 (5^2)}$

15. Given the logarithmic equation $\log_4 A = \log_4 B + \log_4 C$, find A when $B = 16$ and $C = 256$.

Check your answers by turning to the Appendix.

In this activity, you were shown that a convenient way to solve logarithmic equations is by changing them to exponential form and then solving.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

You already know from its definition that $\log_a a^d = d$. You will now evaluate a special case of logarithm.

Example

Evaluate $\log_a a$, where $a > 0$ and $a \neq 1$.

Solution

If $y = \log_a x$ and $x = a$, then $y = \log_a a$. This can be written in exponential form.

$$a^1 = a^y$$

Therefore, $1 = y$.
In other words, $\log_a a = 1$, where $a > 0$.

That is, $\log_2 2 = 1$, $\log_3 3 = 1, \ldots,$ $\log_{10} 10 = 1$, and so on.

This is a very important case. Make sure that you remember the answer.

Well, there certainly are a lot of things to remember when working with logarithmic and exponential forms. You would be well-advised to make sure that you understand this section well before venturing forward.

1. Fill in the blanks for the following statements.

a. The _____ of the exponential function $y = b^x$, where $b > 0$ and $b \neq 1$, is obtained by interchanging the domain and _____ to give _____.

b. The expression $y = \log_b x$ is read as "y is the logarithm of _____ to the base _____." The exponential form _____ is equivalent to $y = \log_b x$.

c. In order to draw the graph of $y = \log_3 x$, change this logarithmic form to the exponential form _____. To prepare a table of values, choose values for _____, then calculate corresponding _____-values. To find the value of $y = \log_3 12$ from the plotted curve, locate 12 on the _____-axis; then draw a _____ line until it intersects the curve. Then draw a _____ line until it meets the _____-axis.

d. The statement $\log 1000 = 3$ means that the base is _____, and this is called the _____ logarithm.

e. The statement $3^5 = 3^x$, means that $x =$ _____; because when bases are equal, the _____ are equal and vice versa.

f. When solving expressions of the form $x = \log_a a^m$, then x has the value of _____.

g. When solving expressions of the form $x = a^{\log_a d}$, then x has the value of _____.

h. The expression $\log_a a$, where $a > 0$ and $a \neq 1$, has the value _____.

2. a. State the inverse of $y = a^x$.

b. Change the inverse from exponential to logarithmic form.

3. Use the graph from Example 1 Activity 2, to answer the following:

a. Find the value of $\log_2 14$.

b. Find x when $y = 1.5$; that is $x = 2^{1.5}$.

4. Express in logarithmic form.

a. $x = 5^y$ **b.** $64 = 4^3$

5. Express in exponential form.

a. $\log_a h = d$ **b.** $\log_7 \frac{1}{49} = -2$

c. $\log_{10} 10 = 1$ **d.** $\log_{10} 0.001 = -3$

6. Solve for the unknown in each problem.

a. $\log_4 x = 2$ **b.** $\log_2 32 = x$

c. $\log_x 216 = 3$ **d.** $\log_8 x = -\frac{2}{3}$

e. $\log_x \frac{1}{64} = -\frac{3}{2}$ **f.** $\log_3 3^{2.5} = x$

g. $\log_5 \sqrt[4]{5} = x$ **h.** $6^{\log_6 \frac{1}{32}} = x$

7. Evaluate the following.

a. $\log_5 \left(\log_7 7^{25} \right) = x$ **b.** $\log_4 \left(3^{\log_3 2} \right) = x$

Use a calculator to answer question 8.

8. Solve the following to two decimal places.

a. $\log_{10} 32$ **b.** $\log_{10} 248$

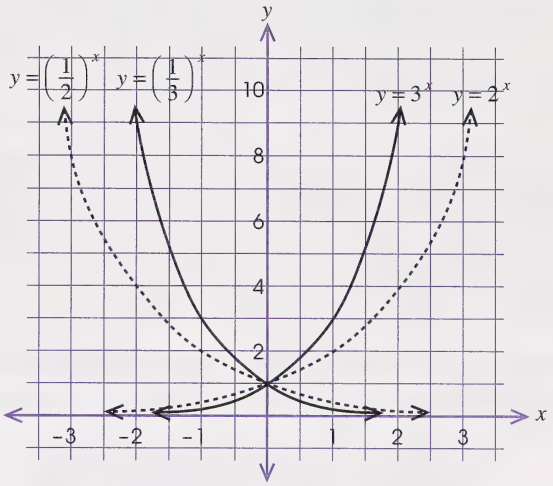
c. $10^{1.08}$ **d.** $10^{0.73}$

Check your answers by turning to the Appendix.

Enrichment

In this section you saw the relationship between exponents and logarithms. You also saw how the graph of a logarithmic function behaves. The following questions will increase your understanding of the exponential and logarithmic graphs.

1. Examine the various graphs of $y = b^x$ for different values of the constant b. Then answer the questions that follow the graph.



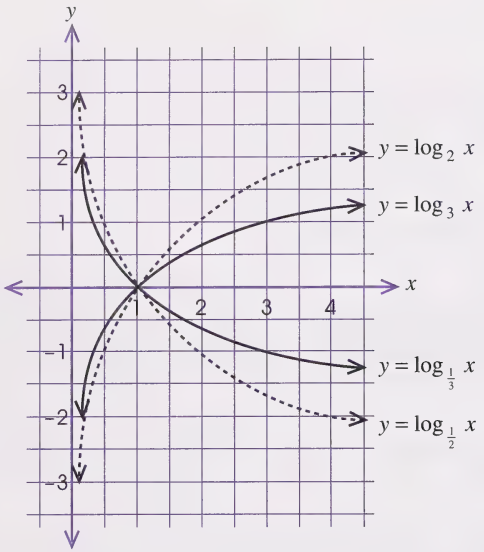
a. For b greater than one and the larger the base, the graph _____ more rapidly from left to right.

b. For $0 < b < 1$ and the smaller the base, the graph _____ more rapidly from left to right.

c. All graphs pass through the point _____.

d. Since $1^x = 1$ for all x, the graph of $y = 1^x$ is the _____ line passing through the point _____.

2. Study the various graphs shown for $y = \log_b x$ for different values of b. Then answer the questions that follow the graph.



a. For b greater than one and the larger the base, the graph increases more _____ from left to right.

b. For $0 < b < 1$ and the larger the base, the graph decreases more _____ from left to right.

c. All graphs pass through the point _____.

Check your answers by turning to the Appendix.

$y = 3^x$

x	−4	−3	−2	−1	0	1	2	3	4
y	$\frac{1}{81}$	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27	81

x	5	6	7	8	9	10
y	243	729	2187	6561	19 683	59 049

From this table of values, note that each number on the top is the $\log_3$ of the number on the bottom. For example, $\log_3 81 = 4$. Notice what happens when two numbers on the right side are multiplied by each other. If you multiply 9 (where $\log_3 9$ is 2) by 27 (where $\log_3 27$ is 3), the product is 243 (where $\log_3 243$ is 5). From this you should be able to interpret that when two numbers on the bottom are multiplied, the corresponding logarithms on the top are added.

This procedure can be used to multiply two numbers on the bottom. For instance, to multiply 81 by 243, add the logarithms from the top. This gives 4 plus 5 equals 9. Locate 9 on the top row of the table and the number opposite 9 is 19 683. Thus, $81 \times 243 = 19\,683$. This method of multiplying numbers implies adding logarithms.

Now check if two numbers from the table can be divided by using their logarithms. For example, if 2187 (for which the logarithm is 7) is divided by 81 (for which the logarithm is 4) the result is 27 (for which the logarithm is $7 - 4 = 3$). The number opposite 3 is 27. The dividing of numbers implies the subtracting of logarithms. In Section 3, you will study the different laws of logarithms in more detail.

Now try the following exercise.

3. Use the previous table of values to calculate the following:

a. $\frac{1}{3} \times 6561$

b. $81 \div \frac{1}{9}$

c. 9^4

d. $\sqrt[3]{19\,683}$

Check your answers by turning to the Appendix.

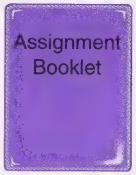
Conclusion

In this section you defined and graphed logarithmic functions, explored relationships between exponential and logarithmic functions, converted statements from logarithmic form to exponential form and vice versa, and solved equations involving logarithms.

Imagine a time before the calculator and the computer. Take your mind back before the development of logarithms by Napier, Briggs, and Bürgi. Try to evaluate a typical expression such as 1.025^{100} that occurs in compound interest. Use ordinary pencil-and-paper techniques. Imagine how long it would take you!

Technology and mathematics have come a long way. In the next section, you will investigate the rules that govern logarithms and that give you the power to evaluate powers, such as 1.025^{100}, in seconds.

Assignment



You are now ready to complete the section assignment.

Section 3: Laws of Logarithms

How are a police officer and a mathematics student alike? Both must apply rules and laws creatively to solve everyday problems. You would not expect effective police officers to be unfamiliar with the regulations that they are expected to enforce. In order to react appropriately in situations that require split-second decision making, a police officer discusses and rehearses procedures repeatedly as part of initial and ongoing training, and does not hesitate to refer to manuals when in doubt.

A good mathematics student studies the rules and practises the procedures of the discipline. Practice, and discussion with others, may clarify difficulties that arise. This makes future problem solving much easier.

In this section you will follow the development and master the laws of logarithms for products, quotients, powers, and roots. You will apply those rules and procedures in problem situations.

Activity 1: Deriving Laws of Logarithms from Exponent Laws

Since the logarithmic function $y = \log_b x$ is the inverse of the exponential function $y = b^x$, you can relate the laws of logarithms to the exponential laws.

To develop the Product Law of Logarithms and the Quotient Law of Logarithms, let $m = b^x$; therefore, $x = \log_b m$. Also, let $n = b^y$; therefore, $y = \log_b n$.

Product Law of Logarithms

Since $m = b^x$ and $n = b^y$, then $mn = \left(b^x\right)\left(b^y\right)$. Therefore, $mn = b^{x+y}$.

Change to log form.

$$\log_b (mn) = \log_b b^{x+y}$$
$$\log_b (mn) = x + y$$

$\log_a a^m$ equals m.

Since $x = \log_b m$ and $y = \log_b n$, $x + y = \log_b m + \log_b n$ and $\log_b (mn) = \log_b m + \log_b n$.

Product Law of Logarithms

$$\log_b (mn) = \log_b m + \log_b n$$

The Product Law of Logarithms indicates that the log of a product is equal to the sum of the log of each factor.

$$\log_b (mn) = \log_b m + \log_b n$$

Conversely, the sum of two logs with the same base is equal to the log of a product.

$$\log_b m + \log_b n = \log_b (mn)$$

You can see how this law works in the following examples.

Example 1

Express $\log_2 36$ using the Product Law of Logarithms.

Solution

$$\log_2 36 = \log_2 (9 \times 4)$$
$$= \log_2 9 + \log_2 4$$

Example 2

Express $\log_2 17 + \log_2 5$ as a log of a product.

Solution

$$\log_2 17 + \log_2 5 = \log_2 (17 \times 5)$$
$$= \log_2 85$$

1. Express each as a sum of logarithms.

   a. $\log_5 (6\times 12)$ **b.** $\log_3 (4\times 16)$

   c. $\log_7 (4\times 11)$ **d.** $\log(3\times 9)$

2. Express each as a log of a product.

   a. $\log_2 12+\log_2 6$ **b.** $\log_4 7+\log_4 11$

   c. $\log_b a+\log_b c$ **d.** $\log_3 2+\log_3 5+\log_3 8$

   e. $\log_x a^2 b+\log_x a^3 b$ **f.** $\log 7.3+\log 18.4$

3. Evaluate the following:

   a. $\log_7 4.9+\log_7 10$ **b.** $\log_2 16+\log_2 4$

   c. $\log_5 1.25+\log_5 500$ **d.** $\log_3 90+\log_3 81+\log_3 0.3$

Check your answers by turning to the Appendix.

Quotient Law of Logarithms

Since $m=b^x$ and $n=b^y$, then $\frac{m}{n}=\frac{b^x}{b^y}$. Therefore, $\frac{m}{n}=b^{x-y}$.

Change to log form.

$$\log_b \frac{m}{n}=\log_b b^{x-y}$$

$$\log_b \frac{m}{n}=x-y$$

Since $x=\log_b m$ and $y=\log_b n$, then $x-y=\log_b m-\log_b n$ and $\log_b \frac{m}{n}=\log_b m-\log_b n$.

Quotient Law of Logarithms

$$\log_b \frac{m}{n}=\log_b m-\log_b n$$

The Quotient Law of Logarithms indicates that the log of a quotient is equal to the difference of the log of the dividend and the log of the divisor.

Conversely, the difference of two logs is equal to the log of the quotient.

You will see how this law applies to the following examples.

Example 1

Express $\log_2\left(\frac{9}{5}\right)$ using the Quotient Law of Logarithms.

Solution

$$\log_2\left(\frac{9}{5}\right)=\log_2 9-\log_2 5$$

Example 2

Express $\log_2 15 - \log_2 3$ as the log of a quotient.

Solution

$$\log_2 15 - \log_2 3 = \log_2 \frac{15}{3}$$
$$= \log_2 5$$

4. Express each as a difference of logarithms.

a. $\log_7 \frac{5.2}{1.7}$ **b.** $\log_5 \frac{8.9}{7.6}$

c. $\log_2 \frac{39}{14}$ **d.** $\log_m \frac{p}{q}$

Check your answers by turning to the Appendix.

What you know at this point is that if the question involves multiplication, then you add logs of the numbers. Also, if the question involves division, then you subtract the logs of the numbers.

Study Example 3 closely.

Example 3

Express $\log_3 \frac{AB}{C}$ as a sum and difference of logarithms.

Solution

$\log_3 \frac{AB}{C} = \log_3 (AB) - \log_3 C$ (Quotient Law)

$\log_3 (AB) = \log_3 A + \log_3 B$ (Product Law)

$\therefore \log_3 \frac{AB}{C} = (\log_3 A + \log_3 B) - \log_3 C$

$= \log_3 A + \log_3 B - \log_3 C$

5. Express each question as a sum or difference of logarithms.

a. $\log_{27} \frac{35.2 \times 1.5}{17.1}$ **b.** $\log_{14} \frac{24 \times 2.7}{18.5}$

c. $\log_b \frac{xy}{z}$ **d.** $\log_c \frac{x}{yz}$

Check your answers by turning to the Appendix.

Example 4

Evaluate $\log_2 10 + \log_2 12.8$.

Solution

Let $\log_2 10 + \log_2 12.8 = x$.

$$\begin{aligned} x &= \log_2 (10 \times 12.8) \\ &= \log_2 128 \\ &= \log_2 2^7 \\ &= 7 \end{aligned}$$

Therefore, $\log_2 10 + \log_2 12.8 = 7$.

Example 5

Solve $\log_5 \frac{25}{625}$.

Solution

$$\begin{aligned} \log_5 \frac{25}{625} &= \log_5 25 - \log_5 625 \\ &= 2 - 4 \qquad \left(\log_5 25 = 2 \text{ and } \log_5 625 = 4\right) \\ &= -2 \end{aligned}$$

Example 6

Given that $\log_7 9 \doteq 1.129$, find the value of $\log_7 63$.

Solution

$$\begin{aligned} \log_7 63 &= \log_7 (9 \times 7) \\ &= \log_7 9 + \log_7 7 \\ &\doteq 1.129 + 1 \qquad (\text{since } \log_7 7 = 1) \\ &\doteq 2.129 \end{aligned}$$

Example 7

Express the following as the logarithm of a product or quotient, and evaluate to the nearest tenth.

$$\log 5.8 + \log 7.9 - \log 2.5$$

Solution

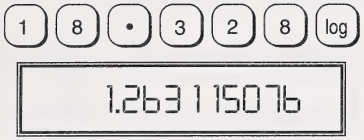
Note that $\log 5.8$, $\log 7.9$, and $\log 2.5$ do not have a stated base. Remember, whenever a base is not stated, assume the base to be 10.

This is called the **common log** and it corresponds to the [log] key on your calculator.

Common logarithm means the base is **10**. If a base is not shown, then it is understood to be 10.

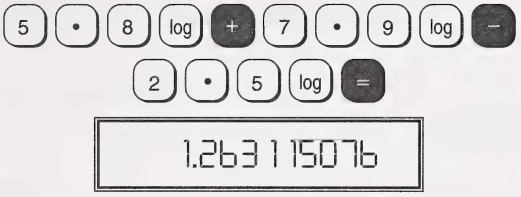
$$\log 5.8 + \log 7.9 - \log 2.5 = \log(5.8 \times 7.9) - \log 2.5$$
$$= \log \frac{5.8 \times 7.9}{2.5}$$
$$= \log 18.328$$

Use a calculator and the log function key to evaluate Example 7.



To the nearest tenth, $\log 18.328 = 1.3$. Therefore, $\log 5.8 + \log 7.9 - \log 2.5 = 1.3$ (to the nearest tenth).

You may also want to try the logarithms individually.



6. Express each as a logarithm of a quotient.

a. $\log_3 27.2 - \log_3 8.1$ **b.** $\log_5 37 - \log_5 146$

7. Express each question as the logarithm of a single number.

a. $\log_x 9.3 + \log_x 0.6 - \log_x 19$
b. $\log 5.8 - \log 60 + \log 40$
c. $\log_4 15 + \log_4 7.4 - \log_4 18$

8. Express each of the following as the logarithm of a product or a quotient; then use your calculator to solve to the nearest tenth.

a. $\log 15.2 - \log 7.6$
b. $\log 26.1 - \log 8.7$
c. $\log 15.1 + \log 7.8 - \log 26.3$
d. $\log 8.9 + \log 5.3 - \log 17.6$

9. Evaluate the following:

a. $\log_2 84 - \log_2 5.25$
b. $\log_3 58.5 - \log_3 6.5$
c. $\log_6 12 + \log_6 9 - \log_6 3$
d. $\log_4 20 + \log_4 16 - \log_4 5$

Check your answers by turning to the Appendix.

There is one further law to study that will assist you in understanding logarithms.

Power Law of Logarithms

To develop the Power Law of Logarithms, you need to recall the following:

$$\begin{aligned} m^n &= (m \times m \times m \times \cdots \times m)(n \text{ factors}) \\ \log_b m^n &= \log_b (m \times m \times m \times \cdots \times m)(n \text{ factors}) \\ &= \log_b m + \log_b m + \log_b m + \cdots + \log_b m (n \text{ factors}) \\ &= n \log_b m \end{aligned}$$

Power Law of Logarithms

$$\log_b m^n = n \log_b m$$

Now a power m^n can be reduced to a multiplication $n \times \log_b m$ using the power law of logarithms.

You can see how this law is applied in this example.

Example 1

Express $\log_2 5^3$ using the Power Law of Logarithms.

Solution

$$\log_2 5^3 = 3 \log_2 5$$

Remember: You can work in both directions where $3 \log_2 5 = \log_2 5^3$. The Power Law of Logarithms has an interesting twist to it.

Recall from you work in previous courses on radicals. It was stated that $\sqrt{x} = x^{\frac{1}{2}}$ and $\sqrt[3]{x^2} = x^{\frac{2}{3}}$.

Therefore, you can express any radical in the form of a power with a rational exponent.

$$\log_b \sqrt[q]{m^p} = \log_b m^{\frac{p}{q}}$$

By using the Power Law of Logarithms, $\log_b m^{\frac{p}{q}} = \frac{p}{q} \log_b m$.

Study this example.

Example 2

Express $\log_2 \sqrt[3]{x^2}$ using the Power Law of Logarithms.

Solution

$$\begin{aligned} \log_2 \sqrt[3]{x^2} &= \log_2 x^{\frac{2}{3}} \\ &= \frac{2}{3} \log_2 x \end{aligned}$$

10. Express each question as a product using the Power Law.

a. $\log_2 3^2$ **b.** $\log_4 5^3$ **c.** $\log_3 5^{\frac{1}{3}}$

d. $\log_6 4^{\frac{2}{3}}$ **e.** $\log_c x^{-1}$ **f.** $\log_a y^{-2}$

g. $\log\sqrt{5}$ **h.** $\log\sqrt[3]{5^4}$ **i.** $\log_b \dfrac{1}{x^2}$

j. $\log_a \dfrac{1}{y^{-3}}$

11. Express each question in the form $\log_b M$.

a. $2\log_3 5$ **b.** $3\log_4 7$ **c.** $\frac{2}{3}\log_9 7$

d. $\frac{3}{5}\log_8 23$ **e.** $n\log_b x$ **f.** $a\log_c d$

g. $\frac{1}{x}\log_p y$ **h.** $\frac{1}{a}\log_b x$

12. Write each question as a single logarithm.

a. $\log M + \log N^2 - \log P$

b. $\log A^2 + \log B - \log C^3$

c. $\log_3 x + \log_3 y - (\log_3 a + \log_3 b)$

d. $\log_2 a + \log_2 b - \log_2 c - \log_2 d$

e. $\log_b 2x + (\log_b x - \log_b y)$

f. $\log_3 4a + \log_3 a - \log_3 c$

g. $\log_a \dfrac{a}{x} - \log_a ax$

h. $\log_b \dfrac{c}{x^2} - \log_b cx^3$

i. If $\log_5 5 = 1$, show that $\log_5 5^n = n$.

13. Given that $\log_3 2 \doteq 0.63$ and $\log_3 5 \doteq 1.465$, use the laws of logarithms to evaluate each of the following questions.

a. $\log_3 4$ **b.** $\log_3 25$ **c.** $\log_3 10$

d. $\log_3 20$ **e.** $\log_3 \frac{5}{2}$ **f.** $\log_3 \frac{2}{5}$

g. $\log_3 0.25$ **h.** $\log_3 0.125$

Check your answers by turning to the Appendix.

You should now have a thorough understanding of the laws of logarithms. These laws are important for expressing logarithms in different forms.

Activity 2: Applying the Laws of Logarithms

In the next two examples, a single logarithmic expression will be expressed in terms of two separate logarithmic expressions.

Example 1

Express $\log_b m^2 n$ in terms of $\log_b m$ and $\log_b n$.

Solution

$$\begin{aligned}\log_b m^2 n &= \log_b m^2 + \log_b n && \text{(Product Law)}\\ &= 2\log_b m + \log_b n && \text{(Power Law)}\end{aligned}$$

Example 2

Express $\log_2 \sqrt[3]{mn}$ in terms of $\log_2 m$ and $\log_2 n$.

Solution

$$\begin{aligned}\log_2 \sqrt[3]{mn} &= \log_2 (mn)^{\frac{1}{3}}\\ &= \frac{1}{3}\log_2 (mn)\\ &= \frac{1}{3}\left(\log_2 m + \log_2 n\right)\end{aligned}$$

In the following example, the logarithm of the root of a number is determined by separating a number into two factors before applying the laws.

Example 3

Given that $\log_3 6 = 1.631$, find the value of $\log_3 \sqrt{18}$.

Solution

$$\begin{aligned}\log_3 \sqrt{18} &= \log_3 (18)^{\frac{1}{2}}\\ &= \frac{1}{2}\log_3 18\\ &= \frac{1}{2}\log_3 (6\times 3)\\ &= \frac{1}{2}\left(\log_3 6 + \log_3 3\right)\\ &\doteq \frac{1}{2}(1.631+1)\\ &\doteq \frac{1}{2}(2.631)\\ &\doteq 1.316\end{aligned}$$

Switch $\log_3 3$ to exponential form.

Therefore, $\log_3 \sqrt{18} \doteq 1.316$.

Now the log of the cube root of a number is determined by using a calculator.

Example 4

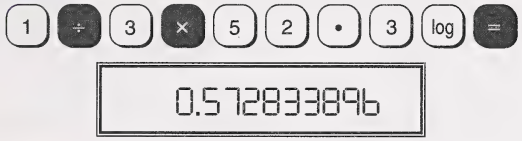
Solve $\log\sqrt[3]{52.3}$.

Solution

$\log x$ is the same as $\log_{10} x$.

$$\log\sqrt[3]{52.3} = \log(52.3)^{\frac{1}{3}}$$
$$= \frac{1}{3}\log 52.3$$

You can evaluate by using a calculator.



Thus, $\log\sqrt[3]{52.3} \doteq 0.573$.

Laws of Logarithms Summary

Product Law: $\log_b (mn) = \log_b m + \log_b n$

Quotient Law: $\log_b \frac{m}{n} = \log_b m - \log_b n$

Power Law: $\log_b m^n = n\log_b m$

$$\log_b \sqrt[q]{m^p} = \log_b m^{\frac{p}{q}}$$
$$= \frac{p}{q}\log_b m$$

1. Evaluate the following to the nearest hundredth.

a. $\log 5^{\frac{1}{2}}$ **b.** $\log 7^{\frac{1}{3}}$ **c.** $\log\sqrt[3]{15.2}$

d. $\log\sqrt[4]{5}$ **e.** $\log\left(5.2^{\frac{2}{3}} \times 3.8^{\frac{1}{5}}\right)$ **f.** $\log\left(6^{\frac{3}{2}} \times 4^{\frac{3}{8}}\right)$

2. Evaluate the following:

a. $\log_3 \sqrt[3]{9}$

b. $\log_4 \left(64 \times \sqrt[5]{4}\right)$

c. $\log_5 25\sqrt{125} + \log_3 \left(81 \times \sqrt[3]{243}\right)$

3. Given that $\log_6 3 \doteq 0.613$, evaluate the following to three decimal places.

a. $\log_6 \sqrt[4]{3}$ **b.** $\log_6 \sqrt[4]{9}$ **c.** $\log_6 108$

4. Write the following as a single logarithm.

   a. $\log A + 3\log B - 4\log C$

   b. $\frac{1}{3}(\log A + \log B - \log C)$

   c. $\frac{1}{4}\log_b A + 5\log_b B - 3\log_b A$

   d. $\frac{1}{7}(\log_5 A + 3\log_5 B) - 2(\log_5 C + \log_5 D)$

   e. $\frac{1}{3}(2\log_2 A - 5\log_2 B - 4\log_2 C)$

   f. $\log_b \sqrt{\frac{c}{y^3}} - \log_b \sqrt{cy}$

Check your answers by turning to the Appendix.

In this activity you discovered the importance of the laws of logarithms for expressing logarithms in different forms or for evaluating them.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

In this section you studied the laws of logarithms and some simple applications. The following exercises will help you to review the laws.

Complete the blanks in the following statements.

1. The Product Law of Logarithms states that
   $\log_b (mn) =$ _____.

2. The Product Law of Logarithms indicates that the log of a product equals the sum of the log of each _____.

3. The Quotient Law of Logarithms states that
   $\log_b \frac{m}{n} =$ _____.

4. The Quotient Law of Logarithms indicates that the log of a quotient equals the _____ of the log of the dividend and the log of the divisor.

5. The Power Law of Logarithms states that
   $\log_b m^n =$ _____.

Do the following problems if you need further practice in this section.

6. Express $\log_2 (5 \times 4)$ as a sum or difference of logarithms.

7. Express $\log_c 20 + \log_c 16 - \log_c 7$ as a product or quotient of logarithms.

8. Express the following as the logarithm of a product or quotient; then use your calculator to evaluate to the nearest hundredth.

$$\log 18.1 + \log 12.4 - \log 19.3$$

9. Given that $\log_5 4 = 0.861$ and $\log_5 6 = 1.113$, use the laws of logarithms to solve $\log_5 24$.

10. Solve $\log_5 10 + \log_5 15 - \log_5 6$.

11. Express $\log_7 3^{\frac{1}{3}}$ as a product.

12. Express $4\log_3 12$ in the form $\log_b M$.

13. Evaluate $\log\left(6^{\frac{1}{3}} \times 5.3^{\frac{2}{3}}\right)$ to the nearest hundredth.

14. Evaluate $\log_7\left(49 \times \sqrt[3]{7}\right)$.

15. Given that $\log_6 2 \doteq 0.387$ and $\log_6 5 \doteq 0.898$, solve $\log_6 \sqrt[4]{20}$ to three decimal places.

16. Write $\frac{1}{3}\left(\log_5 A + 2\log_5 B\right) - 7\log_5 C$ as a single logarithm.

Check your answers by turning to the Appendix.

Enrichment

After studying the laws of logarithms and doing the exercises, you will now see how these laws can be used to solve simple logarithmic equations. In Section 4 you will solve more difficult logarithmic equations.

Example 1

Solve the logarithmic equation for A.

$$\log_6 A = \log_6 4 + \log_6 7$$

Solution

$$\log_6 A = \log_6 (4 \times 7)$$
$$A = 4 \times 7$$
$$= 28$$

Example 2

Solve the logarithmic equation for A.

$$2\log_3 A = 4\log_3 9$$

Solution

$$\log_3 A^2 = \log_3 9^4$$
$$A^2 = 9^4$$
$$A = 9^2$$
$$= 81$$

Example 3

Solve the following logarithmic equation for A.

$$\log_5 A - \log_5 7 = 1$$

Solution

$$\log_5 \left(\frac{A}{7}\right) = 1$$

$$\log_5 \left(\frac{A}{7}\right) = \log_5 5$$

$$\frac{A}{7} = 5$$

$$A = 35$$

1. Solve the following for A.

a. $\log_7 A = \log_7 20 - \log_7 4$

b. $3\log_2 A = 6\log_2 8$

c. $2\log_{24} A - \log_{24} 6 = 1$

2. Given $\log_5 4 \doteq 0.861$ and $\log_5 6 \doteq 1.113$, use the law of logarithms to solve $\log_5 \left(\frac{3}{2}\right)$.

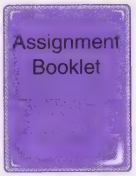
Check your answers by turning to the Appendix.

Conclusion

In this section, you developed the rules for the logarithms of products, quotients, powers, and roots. You applied these rules to write the logarithm of an expression in expanded form, or to combine several logarithms into a single logarithm. Also, you discovered that these rules held regardless of the base of the logarithm.

Just as the courts must apply the laws of the state uniformly and consistently, so you must learn to apply, when warranted, the laws of logarithms consistently and accurately in a variety of problem-solving contexts. The last section of this module will explore the application of logarithms of real-world situations.

Assignment



You are now ready to complete the section assignment.

Section 4: Applications

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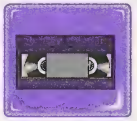
Although considered a scenic and fair-weather state, California is subject to many earthquakes which, in the past, have caused extensive damage and sometimes loss of life. The media often characterize earthquakes by using the Richter scale, which measures the magnitude of an earthquake.

The Richter scale is one example of a logarithmic scale. Other examples of logarithmic scales include the decibel scale (used for measuring the loudness of sounds) and the pH scale (used by chemists to measure the acidity of a chemical solution).

Logarithmic scales are just one application of logarithms. In this section you will apply logarithms to exponential equations, problems involving exponential growth and radioactive decay, and compound interest. You will solve equations containing both common logarithms and logarithms with bases other than 10.



Before starting this section, you may wish to view the video *Graphing Logarithms and Exponents* from the *Graphing Mathematical Concepts* series, ACCESS Network. The video summarizes some of the concepts you have looked at in this module, and presents some practical applications of exponential and logarithmic equations. Keep in mind what you saw on the video as you work through this section. You may find it useful to view parts or all of the video again. This video is available from the Learning Resources Distributing Centre.

Activity 1: Solving Exponential Equations

Do you recall the discussion on exponential equations? You were able to solve equations with variable expressions in the exponent by converting each side of the equation to a power of a common base. In the following examples, you can see an alternate method.

If two expressions are equal, the logarithms of those expressions are also equal.

Example 1

Solve for x.

$$2^x = 8$$

Solution

If you use 2 as the base and take the logarithms of both sides, you get the following:

$$\log_2 2^x = \log_2 8 \qquad \left(\log_2 8 = \log_2 2^3\right)$$
$$x \log_2 2 = 3 \log_2 2 \qquad \text{(Power Law)}$$
$$x \times 1 = 3 \times 1$$
$$x = 3$$

Why should questions be solved using logarithms? Study the following example to answer this question.

Example 2

Solve for x.

$$2^x = 5$$

Solution

You cannot express both sides with the same base as in Example 1.

Again, if you use 2 as the base and take the logarithms of both sides, you get the following:

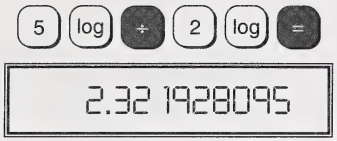
$$\log_2 2^x = \log_2 5$$
Therefore, $x = \log_2 5$

$$\log_a a^m = m$$

This is the value of x that satisfies the equation, but which seldom satisfies the student. You often use **common logarithms** (base of 10) to find an approximate value of x in equations of this type, since there are tables of common logs and your calculators have a log function key.

$$2^x = 5$$
$$\log 2^x = \log 5 \qquad \text{(Use 10 as a base and take the log of each side.)}$$
$$x \log 2 = \log 5 \qquad \text{(Power Law)}$$
$$x = \frac{\log 5}{\log 2} \qquad \text{(Divide both sides by } \log 2.\text{)}$$

Using most calculators, the sequence is as follows:



Thus, $x \doteq 2.322$.

You can now see that if a variable is part of an exponent in an exponential equation, then logarithms to the base 10 may be used in solving the equation. Study the following examples showing how logarithms are used in the solution.

Example 3

Solve $7^{\frac{1}{2}x} = 40$.

Solution

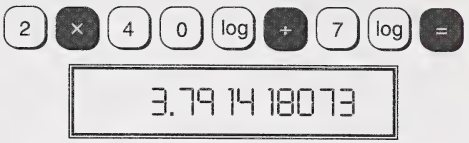
Take the logarithm of both sides.

$$\log 7^{\frac{1}{2}x} = \log(40)$$

$$\frac{1}{2}x\log 7 = \log 40 \qquad \text{(Power Law)}$$

$$x = \frac{2\log 40}{\log 7} \qquad \text{(To obtain } x\text{, multiply both sides by 2 and divide by } \log 7.\text{)}$$

To solve using a continuous calculator sequence, the steps are as follows:



Therefore, $x \doteq 3.791$.

Example 4

Solve $6^{5x} = 3^{2x-1}$.

Solution

Take the logarithm of both sides.

$$\log 6^{5x} = \log 3^{2x-1}$$

$$5x\log 6 = (2x-1)\log 3 \qquad \text{(Power Law)}$$

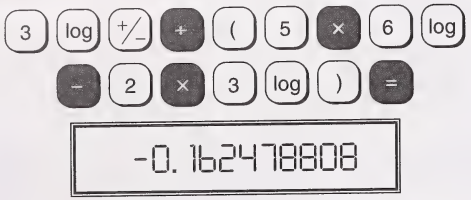
$$5x\log 6 = 2x\log 3 - \log 3 \qquad \text{(distributive property)}$$

$$5x\log 6 - 2x\log 3 = -\log 3 \qquad \text{(terms with unknown variable on LS)}$$

$$x(5\log 6 - 2\log 3) = -\log 3 \qquad \text{(Remove common factor.)}$$

$$x = \frac{-\log 3}{5\log 6 - 2\log 3} \qquad \text{(Divide both sides by } 5\log 6 - 2\log 3.\text{)}$$

The continuous calculator sequence is as follows:



Therefore, $x \doteq -0.162$.

1. Solve the following exponential equations for x.

a. $4^{x} = 9$

b. $7^{x} = 15$

c. $6^{\frac{1}{4}x} = 23$

d. $3^{\frac{1}{5}x} = 14$

e. $3^{2x} = 4^{5x-2}$

f. $5^{x} = 7^{3x-1}$

Check your answers by turning to the Appendix.

When the logarithm has a base other than 10, you may convert it to exponential form and then find the logarithm. The following examples illustrate this.

Example 5

Evaluate $\log_3 7$.

Solution

Let $x = \log_3 7$.

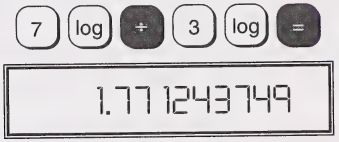
$\therefore\ 3^{x} = 7$ (exponential notation)

$\log\left(3^{x}\right) = \log(7)$ (Take the log of both sides.)

$x \log 3 = \log 7$ (Power Law)

$x = \dfrac{\log 7}{\log 3}$ (Divide both sides by $\log 3$.)

Use a calculator to find the value of x.



Therefore, $\log_3 7 = 1.771$ (to three decimal places).

Note that $x = \log_3 7$ and $x = \frac{\log 7}{\log 3}$ (see previous procedure).

Therefore, $\log_3 7 = \frac{\log 7}{\log 3}$ since $x = x$. This means $\log_3 7 = \frac{\log_{10} 7}{\log_{10} 3}$.

Let $3 = a$, $7 = b$, and $10 = c$. A formula is produced which is the change of base formula. This formula is usually used when $c = 10$.

The change of base formula is $\log_a b = \frac{\log_c b}{\log_c a}$.

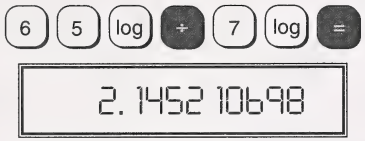
Example 6

Use the change of base formula to find $\log_7 65$ correct to five decimal places.

Solution

$$\log_7 65 = \frac{\log_{10} 65}{\log_{10} 7}$$
$$\doteq \frac{1.812\,913}{0.845\,098}$$
$$\doteq 2.145\,21$$

This is the sequence when you use a calculator.



Therefore, $\log_7 65 \doteq 2.145\,21$ (to five decimal places).

2. Use the change of base formula and solve the following to five decimal places.

a. $\log_4 9$

b. $\log_5 12$

c. $\log_{\frac{1}{5}} 7$

d. $\log_{\frac{1}{4}} 8.6$

e. $\frac{1}{\log_3 8}$

f. $\frac{1}{\log_8 3}$

Check your answers by turning to the Appendix.

In order to solve exponential equations, you had to convert them to logarithms and then solve. Logarithms with different bases can also be simplified by first converting them to base 10.

Activity 2: Solving Logarithmic Equations

The questions you have been studying to this point are characterized as exponential equations, which use the laws of logarithms to solve. In the following examples, the unknown will not be part of the exponent.

Example 1

Solve $x^{\frac{3}{4}} = 15$, and state the answer to three decimal places.

Solution

Take the logarithm of both sides.

$$\log x^{\frac{3}{4}} = \log 15$$

$$\frac{3}{4}\log x = \log 15 \qquad \text{(Power Law)}$$

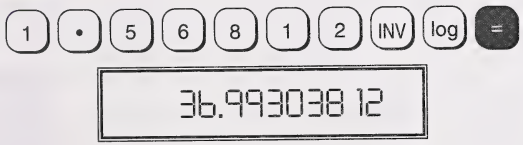
$$\log x = \frac{4}{3}\log 15 \qquad \left(\text{Multiply by } \tfrac{4}{3}.\right)$$

$$\log x \doteq \frac{4}{3}(1.176\,09)$$

$$\log x \doteq 1.568\,12$$

$\log_{10} x \doteq 1.568\,12$ is equivalent to $10^{1.568\,12} \doteq x$.

Use a calculator to calculate x.



Thus, $x \doteq 36.993$ (to three decimal places).

Example 2

Solve $\log_2 x - \log_2 5 = \log_2 3$.

Solution

You must solve for the variable x in the equation. Use the laws of logarithms previously developed to simplify and solve the equation.

$$\log_2 x - \log_2 5 = \log_2 3$$

$$\log_2 \frac{x}{5} = \log_2 3$$

$$\frac{x}{5} = 3$$

$$x = 15$$

$\frac{x}{5} = 3$ is true because both sides are equal and the logarithms are to the base 2.

Verification

Substitute $x = 15$ into the original equation to see if this solution is true.

LS	RS
$\log_2 x - \log_2 5$	$\log_2 3$
$= \log_2 15 - \log_2 5$	
$= \log_2 \frac{15}{5}$	
$= \log_2 3$	

LS = RS

Example 3

Solve $\log_4(2x-1)+\log_4(x+3)=1$ for x.

Solution

$$\log_4(2x-1)+\log_4(x+3)=1$$

$$\log_4(2x-1)(x+3)=1 \qquad \text{(Product Law)}$$

$$\log_4(2x-1)(x+3)=\log_4 4 \qquad (\log_a a=1)$$

$$(2x-1)(x+3)=4 \qquad (\text{If } \log_x a=\log_x b, \text{ then } a=b.)$$

$$2x^2+5x-3=4$$

$$2x^2+5x-7=0$$

$$(2x+7)(x-1)=0$$

$$2x+7=0 \quad \text{or} \quad x-1=0$$

$$x=-\frac{7}{2} \qquad x=1$$

Verification

If you substitute $x=-\frac{7}{2}$ in the original equation, the left side becomes $\log_4(-8)+\log_4(-0.5)$. Since the logarithm of a negative number is not defined, $x=-\frac{7}{2}$ is not admissible.

When $x=1$, $\text{LS}=\log_4(2-1)+\log_4 4$ and $\text{RS}=1$.

LS	RS
$\log_4 1+\log_4 4$	1
$=0+1$	
$=1$	

LS = RS

Since LS = RS, $x=1$ is a solution.

Example 4

Solve $\log x-\log 5=\log 6x+\log 4x$ for x.

Solution

$$\log x-\log 5=\log(6x)+\log(4x)$$

$$\log\frac{x}{5}=\log(6x\times 4x) \qquad \text{(Quotient Law and Product Law)}$$

$$\frac{x}{5}=24x^2$$

$$120x^2=x$$

$$120x^2-x=0$$

$$x(120x-1)=0$$

$$x=0 \quad \text{or} \quad 120x-1=0$$

$$x=\frac{1}{120}$$

Since $\log x$ has a domain greater than zero, the solution is $x=\frac{1}{120}$.

Example 5

Solve $\log_4 4x - \log_4 (x-2) = 3$ for x.

Solution

$$\log_4 4x - \log_4 (x-2) = 3$$

$$\log_4 \frac{4x}{x-2} = 3$$

$$\frac{4x}{x-2} = 4^3$$

$$4x = 64(x-2)$$

$$4x = 64x - 128$$

$$x = \frac{128}{60}$$

$$= \frac{32}{15}$$

1. a. $x^{\frac{2}{5}} = 32.6$ **b.** $x^{\frac{5}{8}} = 27.4$

2. a. $\log_3 x + \log_3 6 = \log_3 5$ **b.** $\log_5 12 - \log_5 x = \log_5 8$

3. a. $\log_5 2x - \log_5 7 = 1$ **b.** $\log_3 3x + \log_3 6 = 1$

4. a. $\log_2 2x + \log_2 (x-1) = 2$

b. $\log_4 2x - \log_4 (2x-1) = 2$

5. a. $\log 6 - \log x = \log 3x + \log 2x$

b. $\log 8x - \log 3x = \log 4 + \log x$

6. a. $3\log x = \log 8$ **b.** $4\log x = \log \frac{1}{16}$

Check your answers by turning to the Appendix.

Activity 3: Applying Exponential and Logarithmic Functions

One of the most common uses of exponential and logarithmic functions is solving problems involving population growth, radioactive decay, and compound interest.

A formula for predicting population growth is $G = G_0\left(2^{cy}\right)$, where G is the population at time y in years, G_0 is the population when $y = 0$, and c is a constant that is always positive.

Look at the next example to see how this formula can be used to predict the population in future years.

Example 1

The population of an urban area was 35 700 in 1975 and 43 600 in 1985. Predict what the population will be in the year 2000.

Solution

Assume exponential population growth, and use the formula $G = G_0\left(2^{cy}\right)$.

To predict G for the year 2000, the values for G_0, c, and y must be known.

Let $y = 0$ in 1975; then $G_0 = 35\,700$. For the year 2000, $y = 2000 - 1975 = 25$. But a value of c must be determined.

To obtain c, use the information given for 1985. That is, $G = 43\,600$, $G_0 = 35\,700$, and $y = 1985 - 1975 = 10$. Substitute these values in the formula as shown.

$$G = G_0\left(2^{cy}\right)$$
$$43\,600 = 35\,700\left(2^{c(10)}\right)$$
$$\frac{43\,600}{35\,700} = 2^{10c}$$
$$\log\frac{43\,600}{35\,700} = 10c\log 2$$
$$c = \frac{\log\frac{43\,600}{35\,700}}{10\log 2}$$
$$\doteq \frac{0.086\,818}{3.010\,29}$$
$$\doteq 0.029$$

For the year 2000, $G_0 = 35\,700$, $c \doteq 0.029$, and $y = 25$.

$$G \doteq 35\,700\left(2^{(0.029)(25)}\right)$$
$$\doteq 59\,009$$

In the year 2000, the population will be approximately 59 000.

The decaying process of radioactive substances is exponential. Half-life is the rate of the decay of a radioactive substance. Recall from Section 1 that half-life is the time required for a substance to decay to one half of its original amount.

The formula $a = a_0\left(2^{-\frac{t}{h}}\right)$ represents the amount a remaining after time t. The original amount is a_0, and h is the half-life of the radioactive substance.

Example 2

A radioactive substance has a half-life of 6 d. What is the time required for 300 g of this substance to decay to 15 g?

Solution

Substitute $a_0 = 300$ g, $a = 15$ g, and $h = 6$ into the formula.

$$a = a_0\left(2^{-\frac{t}{h}}\right)$$

$$a = a_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$$

$$15 = 300\left(\frac{1}{2}\right)^{\frac{t}{6}}$$

$$\frac{15}{300} = \left(\frac{1}{2}\right)^{\frac{t}{6}}$$

$$\log\frac{15}{300} = \log\left(\frac{1}{2}\right)^{\frac{t}{6}}$$

$$\log\frac{1}{20} = \frac{t}{6}\log\frac{1}{2}$$

$$\therefore\ t = \frac{6\log\frac{1}{20}}{\log\frac{1}{2}} \qquad \left(\text{Multiply by 6 and divide by } \log\tfrac{1}{2}.\right)$$

$$\doteq \frac{6(-1.301\,03)}{-0.301\,03}$$

$$\doteq 25.93$$

$$2^{-\frac{t}{h}} = \frac{1}{2^{\frac{t}{h}}} = \left(\frac{1}{2}\right)^{\frac{t}{h}}$$

It takes approximately 26 d for this substance to decay to 15 g.

Under favourable breeding conditions, the population of some organisms can double, or multiply even faster. Bacteria of a certain type are known to divide every hour. The formula $A = A_0\left(2^{\frac{t}{p}}\right)$ represents the number A of bacteria present after t hours. A_0 is the number of bacteria at the beginning, and p is the doubling period.

Example 3

A biologist makes a sample count of bacteria in a culture and finds that it doubles every 4 h. If the culture has 700 bacteria, how many bacteria are in the culture after 25 h?

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Solution

Substitute $A_0 = 700$, $t = 25$, and $p = 4$ into the formula.

$$A = A_0\left(2^{\frac{t}{p}}\right)$$

$$A = 700\left(2^{\frac{25}{4}}\right)$$

$$\log A = \log\left[700\left(2^{\frac{25}{4}}\right)\right]$$

$$\log A = \log 700 + \frac{25}{4}\log 2$$

$$\log A \doteq 2.845\,098 + 1.881\,437$$

$$\log A \doteq 4.726\,535$$

$$A \doteq 53\,276$$

When the interest on the money is added to the principal at regular intervals of time, the interest will also earn interest. This compound interest is represented as an exponential function.

The compound interest formula is $A = P\left(1+\frac{r}{n}\right)^{nt}$, where P is the original amount deposited at a yearly rate of interest r and compounded n times per year. This exponential growth will produce the value A after t years.

Example 4

Find the time period (to the nearest year) required for \$7000 invested at 10% per year and compounded semiannually to grow to \$10 000.

Solution

Substitute $A = 10\,000$, $P = 7000$, $r = 0.10$, and $n = 2$ into the formula.

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$10\,000 = 7000\left(1+\frac{0.10}{2}\right)^{2t}$$

$$\frac{10}{7} = (1.05)^{2t}$$

$$\log\frac{10}{7} = \log(1.05)^{2t} \qquad \text{(Take logs of both sides.)}$$

$$\log\frac{10}{7} = 2t\log(1.05) \qquad \text{(Power Law)}$$

$$t = \frac{\log\frac{10}{7}}{2\log(1.05)}$$

$$\doteq \frac{0.154\,90}{0.042\,38}$$

$$\doteq 3.66$$

The time required is approximately 4 years.

Logarithms are also used to determine the pH of a solution. The pH of a solution is expressed by the formula $\text{pH} = \log_{10}\frac{1}{\text{H}_c}$, where H_c is the hydrogen ion concentration (in mol/L).

Example 5

Determine the pH of a solution which has an H_c of 2.7×10^{-4} mol/l. Specify whether the solution is neutral, alkaline, or acidic according to the following guidelines. If the $pH = 7$, the solution is neutral; if the $pH > 7$, the solution is alkaline; and if the $pH < 7$, the solution is acidic.

Solution

$$pH = \log_{10}\frac{1}{H_c}$$

$$= \log_{10}\frac{1}{2.7\times10^{-4}}$$

$$= \log_{10}1 - \log_{10}\left(2.7\times10^{-4}\right)$$

$$\doteq 0-(-3.569)$$

$$\doteq 3.569$$

The pH of the solution is approximately 3.569; therefore, the solution is acidic.

Logarithms are also used in determining the loudness of sound. The loudness of sound can be calculated using the formula $L_{dB} = 10\log_{10}\frac{I_m}{I_r}$, where L_{dB} is the loudness in decibels (dB), I_m is the intensity of sound being measured, and I_r is the intensity of sound as a reference (that is, minimum intensity at the human hearing level).

Example 6

The sound of a busy street is measured at 80 dB. The sound of ordinary conversation is measured at 40 dB. How many times louder is the sound of the busy street compared to a conversation?

Solution

Busy Street:

$$L_{dB} = 10\log_{10}\frac{I_m}{I_r}$$

$$80 = 10\log_{10}\frac{I_m}{I_r}$$

$$8 = \log_{10}\frac{I_m}{I_r} \quad \text{(Divide by ten.)}$$

$$10^8 = \frac{I_m}{I_r} \quad \text{(exponential form)}$$

$$\therefore I_m = I_r\times10^8$$

Ordinary Conversation:

$$L_{dB} = 10\log_{10}\frac{I_m}{I_r}$$

$$40 = 10\log_{10}\frac{I_m}{I_r}$$

$$4 = \log_{10}\frac{I_m}{I_r}$$

$$10^4 = \frac{I_m}{I_r}$$

$$I_m = I_r \times 10^4$$

$$\therefore \frac{I_m \text{ street}}{I_m \text{ talking}} = \frac{I_r\left(10^8\right)}{I_r\left(10^4\right)}$$

$$= 10^4$$

$$= 10\,000$$

The street sound is 10 000 times louder than an ordinary conversation.

A logarithmic scale is used for comparing intensities of earthquakes. The Ritcher Scale uses the formula $\log_{10}\frac{I_m}{I_r}$, where I_m is the intensity of the earthquake being measured and I_r is the intensity of a reference earthquake.

Example 7

On the Richter Scale, Earthquake A has a magnitude of 7.8, and Earthquake B has a magnitude of 5.4. How many times more intense is Earthquake A compared to Earthquake B?

Solution

Earthquake A:

$$7.8 = \log_{10}\frac{I_m}{I_r}$$

$$10^{7.8} = \frac{I_m}{I_r} \quad \text{(exponential form)}$$

$$I_m = 10^{7.8}\, I_r$$

Earthquake B:

$$5.4 = \log_{10}\frac{I_m}{I_r}$$

$$10^{5.4} = \frac{I_m}{I_r}$$

$$I_m = 10^{5.4}\, I_r$$

$$\therefore \frac{I_m \text{ for Earthquake A}}{I_m \text{ for Earthquake B}} = \frac{10^{7.8}\, I_r}{10^{5.4}\, I_r}$$

$$= 10^{7.8-5.4}$$

$$= 10^{2.4}$$

$$\doteq 251$$

Earthquake A is about 251 times more intense than Earthquake B.

1. The population of a country was 7.86×10^7 in 1970 and 8.03×10^7 in 1980. Predict the population for the year 2000.

2. A radioactive substance has a half-life of 87 d. Find the original amount of this substance if 28 g remain after 62 d.

3. Find the total amount for the following:

   a. If $1200 is compounded quarterly for six years at 9% per annum.

   b. If $800 is invested for four years at 10% per annum and compounded semiannually.

4. **a.** Calculate the number of years required for $800 to double if it is invested at 8% per annum, compounded quarterly.

   b. Determine the number of years required for $700 to triple in size if it is invested at 10% per annum, compounded semiannually.

5. **a.** The pH of a solution is 7.5. Determine the hydrogen ion concentration in mol/L.

   b. Calculate the hydrogen ion concentration in mol/L for a solution that has a pH of 5.6.

6. **a.** How many times louder is a sound at the 50 dB level compared to a sound at the 25 dB level?

   b. A gas lawnmower with a defective muffler has a loudness level of 108 dB. After a new muffler is installed the loudness level is 94 dB. How many time louder was the lawnmower before the muffler was repaired?

7. **a.** Earthquake A has a magnitude of 6.5 on the Richter Scale, and Earthquake B has a reading of 4.7. How many times greater is Earthquake A compared to Earthquake B?

   b. How many times greater is Earthquake C compared to Earthquake D if their respective Richter readings are 5.9 and 5.1?

8. A bacteria culture triples every 4 h and starts with 2000 bacteria. How long will it take for the population to reach 27 000?

Check your answers by turning to the Appendix.

You had an opportunity to see the connection of logarithmic and exponential functions to real-world situations.

Follow-up Activities

If you had difficulties understanding the concepts in the activities, it is recommended that you do the Extra Help. If you have a clear understanding of the concepts, it is recommended that you do the Enrichment. You may decide to do both.

Extra Help

In this section you noticed that logs can be written with different bases. You also learned how to solve real-life problems. The following exercises will help you further understand the laws that have been applied throughout this section.

For questions 1 to 11, complete the statements.

1. Steps in solving $3^x = 7$ are as follows:

$\log 3^x = \log 7$ (Take base _____ log of each side.)

$x \log 3 = \log 7$ (_____ Law)

$x = \dfrac{\log 7}{\log 3}$ (_____ both sides by $\log 3$.)

$\doteq$ _____ (to three decimal places)

2. a. The formula $\log_a b = \dfrac{\log_c b}{\log_c a}$ is called the _____ formula.

b. Use this formula to evaluate $\log_4 9$.

$\log_4 9 = \dfrac{\log_{10} 9}{\square}$

$\log_4 9 \doteq$ _____ (to five decimal places)

3. Steps in solving $x^{\frac{3}{7}} = 9$ are as follows:

$\log x^{\frac{3}{7}} = \log 9$

_____ $\log x = \log 9$ (Power Law)

$\log x =$ _____ $\log 9$

$\log x \doteq$ _____ (to six decimal places)

$x \doteq$ _____ (to four decimal places)

4. Steps for solving $\log_3 5x - \log_3 (2x-3) = 2$ are as follows:

$\log_3 \left(\dfrac{5x}{2x-3}\right) = 2$ (_____ Law)

$\dfrac{5x}{2x-3} =$ _____ (exponential form)

$5x =$ _____ $(2x-3)$

$x =$ _____

5. Common logarithms use base _____.

6. The formula $G = G_0\left(2^{cy}\right)$ is used for predicting _____.

7. The formula $a = a_0\left(2^{-\frac{t}{h}}\right)$ represents the amount remaining in the decaying process of radioactive substances. In the formula, h represents the _____ of the substance.

8. The formula $A = P\left(1+\frac{r}{n}\right)^{nt}$ is the _____ formula.

9. In the formula $\text{pH} = \log_{10}\frac{1}{\text{H}_c}$, the H_c represents the _____ in mol/L.

10. The loudness of sound is measured in _____. The formula is $L = 10\log_{10}\frac{I_m}{I_r}$, where I_m is the intensity of sound being _____.

11. The _____ scale uses the formula $\log_{10}\frac{I_m}{I_r}$ to compare the intensities of _____.

12. Solve for x.

    a. $3^{4x} = 5^{2x+1}$

    b. $\log_4(3x-1)+\log_4(x+1)=1$

    c. $\log_3 2x - \log_3(2x-1) = 2$

13. A radioactive substance has a half-life of 28 years. Calculate the time required for 40 g to decay to 4 g.

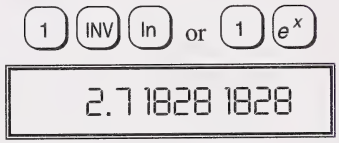
Check your answers by turning to the Appendix.

Enrichment

Common logarithms have been used to solve for x in the preceding examples and exercises. There is a system called **natural logarithms** which uses the base ***e***. The value of base e is approximately 2.718 281 828. Thus, $\log_e x$ means the natural logarithm of x and is often written in the form $\ln x$.

If $\log_e x = 1$, then the equivalent exponential form is $e^1 = x$.

Now use a calculator to calculate x.



Thus, x is 2.718 281 828; but since $e^1 = x$, $e \doteq 2.718\,281\,828$.

If you press the keys [1] [INV] [ln] [ln] your calculator should show 1.

The sequence [1] [INV] [ln] gives the approximate value for e.

When you press [ln] a second time you are obtaining the answer for $\log_e e$ which is 1. For example $\ln e = 1$.

Study the following examples containing natural logarithms.

Use a calculator to answer Examples 1, 2, and 3.

Example 1

Find the natural logarithm of 7; then write the result in exponential form.

Solution

$\log_e 7 = \ln 7$.

[7] [ln]

1.9459 10 149

The natural log of 7 is about 1.945 910 149.

If $\log_e 7 = x$, the exponential form is $e^x = 7$. Thus, $e^{1.945\,910\,149} \doteq 7$.

Example 2

Solve $e^x = 6.8$ for x.

Solution

Take the natural logarithm of both sides.

$$\ln e^x = \ln 6.8$$
$$x \ln e = \ln 6.8 \quad \text{(Power Law)}$$
$$x(1) = \ln 6.8 \quad (\ln e = 1 \text{ or } \log_e e = 1.)$$

[6] [•] [8] [ln]

1.9 169226 12

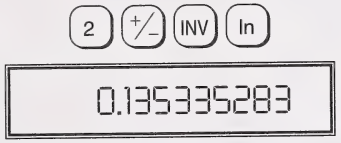
Thus, $x \doteq 1.916\,922\,612$.

Example 3

For the natural logarithmic equation $\ln x = -2$, solve for x.

Solution

$\log_e x = -2$ is equivalent to $e^{-2} = x$



Thus, $x \doteq 0.135\,335\,283$.

For a check after obtaining $0.135\,335\,283$, press ln and you should get -2. That is, $\ln 0.135\,335\,283 = -2$.

Use a calculator to answer questions 1 and 2.

1. Find the natural logarithm for each of the following to four decimal places.

a. $\log_e 8$ **b.** $\log_e 25$ **c.** $\ln 12$
d. $\ln 2$ **e.** $\ln 0.8$ **f.** $\ln 0.005$

2. Calculate each logarithm to six decimal places; then write each expression in exponential form.

a. $\ln 6$ **b.** $\ln 17$ **c.** $\ln 0.3$ **d.** $\ln 0.008$

3. Solve for x, and express your answers to six decimal places.

a. $e^x = 18$ **b.** $e^x = 27$
c. $e^x = 0.49$ **d.** $e^x = 0.048$

4. Solve the following natural logarithmic equations.

a. $\ln x = 4$ **b.** $\ln x = 2.5$
c. $\ln x = -1.8$ **d.** $\ln x = -2.7$

Check your answers by turning to the Appendix.

If you did not have access to a set of natural logarithm tables or a calculator, then you could calculate the approximate natural logarithm of any number by using the formula.

$$\ln N = \frac{\log_{10} N}{\log_{10} e}$$
$$= \frac{\log N}{0.4343}$$
$$= 2.303 \log N$$

To show how this formula is obtained, let $x = \ln N$.

$$\therefore N = e^x \qquad \text{(exponential form)}$$

$$\log N = \log e^x \qquad \text{(Take the common log of both sides.)}$$

$$\log N = x \log e \qquad \text{(Power Law)}$$

$$x = \frac{\log N}{\log e} \qquad \text{(Divide by } \log e.\text{)}$$

$$\ln N = \frac{\log N}{\log e}$$

$$= \frac{\log N}{0.4343}$$

$$= 2.303 \log N$$

Example 4

Use the formula $\ln N = 2.303 \log N$ to find the natural logarithm of 8.4.

Solution

$$\ln N = 2.303 \log N$$

$$\ln 8.4 = 2.303 \log 8.4$$

$$\ln 8.4 \doteq 2.303(0.9243)$$

$$\ln 8.4 \doteq 2.129$$

Example 5

Use the logarithm laws to calculate the numerical value of $\ln \frac{1}{5}$.

Solution

$$\ln \frac{1}{5} = \ln 1 - \ln 5$$

$$\doteq 0 - 1.609\,44 \qquad \left(\ln 1 = 0, \text{ since } e^0 = 1.\right)$$

$$\doteq -1.609\,44$$

Example 6

Use the logarithm laws to calculate the numerical value of $\ln 15$ given $\ln 3 \doteq 1.098\,61$ and $\ln 5 \doteq 1.609\,44$.

Solution

$$\ln 15 = \ln(3 \times 5)$$

$$= \ln 3 + \ln 5$$

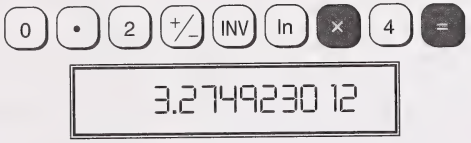
$$\doteq 1.098\,61 + 1.609\,44$$

$$\doteq 2.708\,05$$

Example 7

Use a calculator to find the value of $4e^{-0.2}$. Find the value of $e^{-0.2}$ first; then multiply by 4.

Solution



Thus, $4e^{-0.2} \doteq 3.2749$ (to four decimal places).

Example 8

Find the value of $4e^{-0.2}$ by using the laws of natural logarithms.

Solution

Let $x = 4e^{-0.2}$.

$\ln x = \ln 4e^{-0.2}$ (Take the natural log of both sides.)

$\ln x = \ln 4 + (-0.2)\ln e$ (Product and Power Laws)

$\ln x \doteq 1.386\,29 + (-0.2)1$ $\left(\log_e e = 1\right)$

$\ln x \doteq 1.186\,29$

$x \doteq 3.2749$

Thus, $4e^{-0.2} \doteq 3.2749$.

The following theory will show the advantage to using base e in calculations.

In New York City, the population was about 7.07 million in 1980, and the population was increasing at 0.35% per year. The formula $P = 7.07(1.0035)^t$ shows the population P in millions dependent on t ,where t is the number of years from 1980. To use this formula with base e instead of base 10, you must express 1.0035 as an exponent of e. Let $1.0035 = e^h$; thus, $h = \ln 1.0035$ by definition. Use a calculator and press the keys 1.0035 [ln]. The display should read 0.003 493 889. The value of h to two significant digits is 0.0035. Thus, $1.0035 \doteq e^{0.0035}$, and the formula $P = 7.07(1.0035)^t$ can be changed to $P = 7.07e^{0.0035t}$.

Note that the constant in the exponent of e is 0.0035, which is the growth rate that was given. When an exponential function is expressed with the natural base e, the constant in the exponent of e is the rate of growth. This shows the advantage of using the natural base in exponential growth and decay problems.

Example 9

The population of a country was 6 million in 1984, and the rate of growth was approximately 1.8% per year.

- Determine a formula for the population P in million, t years after 1984. Then use this formula to calculate the population in 1994.
- Calculate the number of years from 1984 until the population is 9 million.

Solution

The formula is $P = 6e^{0.018t}$.

For 1994, the time period t is ten years from 1984.

$$\begin{aligned} P &= 6e^{0.018(10)} \\ &= 6e^{0.18} \\ &\doteq 6(1.197\,217\,363) \\ &\doteq 7.183\,304 \end{aligned}$$

In 1994, the population will be approximately 7.183 million.

To calculate the number of years when the population is 9 million, substitute 9 for P and take the natural logarithm of each side.

$$\begin{aligned} 9 &= 6e^{0.018t} \\ \ln 9 &= \ln 6e^{0.018t} \\ \ln 9 &= \ln 6 + 0.018t \ln e \\ \ln 9 &= \ln 6 + 0.018t \qquad \ln e = 1, \text{ since } \log e^{e} = 1. \\ t &= \frac{\ln 9 - \ln 6}{0.018} \\ &\doteq \frac{2.197\,224\,577 - 1.791\,759\,469}{0.018} \\ &\doteq 22.5 \end{aligned}$$

In about 22.5 years from 1984, the population will be 9 million.

5. Use the change of base formula to calculate $\log_8 8\,657\,432$ to four decimal places.

6. If $\log_a 8 = 7$, show that $\log_8 a = \frac{1}{7}$.

7. **a.** Show $(\log_c d)(\log_d c) = 1$, where c and d are positive and not equal to one.

b. If $\log_a b = \frac{1}{s^2}$ and $\log_b a = 2s^4$, show $s = \pm\frac{\sqrt{2}}{2}$.

8. Use the formula $\ln N \doteq 2.303 \log N$ to find the natural logarithm of 15.7 to three decimal places.

9. Use the laws of logarithms to calculate numerical values for the following if $\ln 2 \doteq 0.6931$ and $\ln 3 \doteq 1.0986$.

a. $\ln \frac{1}{6}$ **b.** $\ln 12$

10. Find the value of $6e^{-0.4}$ by using the laws of natural logarithms.

11. The population of a country was 18.4 million in 1980 and the rate of growth was 1.2% per year.

a. Determine a formula for the population, then use this formula to calculate the population in 1996.

b. Determine the number of years from 1980 until the population is 25 million.

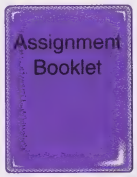
Check your answers by turning to the Appendix.

Conclusion

In this section you applied logarithms to exponential equations and problems involving exponential growth, radioactive decay, and compound interest. You solved equations containing both common logarithms and logarithms with bases other than 10. You also investigated logarithmic scales such as the decibel scale (used for measuring the intensity of sound) and the Richter scale (used for measuring the intensity of seismic activity like earthquakes).

Did you know that four out of every five earthquakes in the world occur along the boundaries of the Pacific Ocean? Although, for North America, this region includes British Columbia, the most devastating earthquakes in recent times have occurred in Mexico, California, and Alaska.

Assignment



You are now ready to complete the section assignment.

Module Summary

In this module you were first introduced to the exponential function. Next, you were introduced to the logarithmic function as the inverse of the exponential function. This analysis of the exponential function and the logarithmic function is an application of the general study of functions from Mathematics 20.

Once the logarithmic function was defined, you studied the laws governing logarithmic and some of their applications to solving equations and to the sciences. One of those applications was the exponential growth of populations.

Populations cannot grow exponentially for an indefinite period of time. There are limits to all growth. Eventually, a population stabilizes or, because of uncontrollable external forces, collapses. For instance, the once great herds of the African savannah have all but vanished. There are other populations, both animal and human, that recognize that population size approaches an equilibrium at best. If you take Mathematics 31, you will encounter other models that are based on logarithms as well.

Final Module Assignment

Assignment Booklet

You are now ready to complete the final module assignment.

APPENDIX

Glossary

Suggested Answers

Glossary

Common logarithms: logarithms which use 10 as a base

Doubling period: the period of time required for a given amount of bacteria to double

Exponential function: a function of the form $y = a^x$, where $a > 0$ and $a \neq 1$

Half-life: the period of time during which a given amount of a radioactive element decays until one-half of it is left

Logarithm: (of a positive number) the exponent that indicates the power to which a number is raised to produce a given number

Natural logarithms: logarithms which use the base $e = 2.718\,28\ldots$

Vertical line test: any vertical line that intersects the graph at one point indicates a function

Suggested Answers

Section 1: Activity 1

1. a. $\left(3^2\right)^2 \div \left(3^3 \times 3^2\right) = 3^4 \div 3^5$

$= 3^{4-5}$

$= 3^{-1}$

$= \frac{1}{3^1}$ or $\frac{1}{3}$

b. $\frac{\left(x^2yz\right)^2}{\left(xyz^2\right)^3} \times \left(\frac{2}{y}\right)^0 = \frac{x^4y^2z^2}{x^3y^3z^6}$

$= \frac{x}{yz^4}$

c. $\left(\frac{x}{y}\right)^2\left(\frac{y}{z}\right)^3\left(\frac{y}{x}\right)^2 = \left(\frac{x^2}{y^2}\right)\left(\frac{y^3}{z^3}\right)\left(\frac{y^2}{x^2}\right)$

$= \frac{y^3}{z^3}$

d. $\frac{\left(t^{x-3}\right)\left(t^{2x+5}\right)}{t^2} = \frac{t^{3x+2}}{t^2}$

$= t^{3x}$

e. $3^{\frac{1}{7}x} \times 27^{\frac{2}{7}x} \div 9^{2x-1} = \dfrac{3^{\frac{1}{7}x} \times \left(3^3\right)^{\frac{2}{7}x}}{\left(3^2\right)^{2x-1}}$

$= \dfrac{3^{\frac{1}{7}x+\frac{6}{7}x}}{3^{4x-2}}$

$= \dfrac{3^{1x}}{3^{4x-2}}$

$= 3^{2-3x}$

f. $\sqrt{8} + \sqrt[3]{32} = \sqrt{2^3} + \sqrt[3]{2^5}$

$= 2^{\frac{3}{2}} + 2^{\frac{5}{3}}$

g. $\sqrt[6]{4^3} \div \sqrt[6]{8^2} = 4^{\frac{3}{6}} \div 8^{\frac{2}{6}}$

$= 4^{\frac{1}{2}} \div 8^{\frac{1}{3}}$

$= \left(2^2\right)^{\frac{1}{2}} \div \left(2^3\right)^{\frac{1}{3}}$

$= 2 \div 2$

$= 1$

h. $\dfrac{a^{10} + a^9}{a^2 + a} = \dfrac{a^9(a+1)}{a(a+1)}$

$= \dfrac{a^9}{a}$

$= a^8$

2. a.

LS	RS
$\dfrac{\left(3^{n+1}\right)\left(3^{n+2}\right)(3)}{9^n}$	81
$= \dfrac{3^{n+1+n+2+1}}{\left(3^2\right)^n}$	
$= \dfrac{3^{2n+4}}{3^{2n}}$	
$= 3^4$	
$= 81$	

LS = RS

b.

LS	RS
$\dfrac{5^{4n} + 5^{3n}}{5^n + 5^{2n}}$	5^{2n}
$= \dfrac{5^{3n}\left(5^n + 1\right)}{5^n\left(1 + 5^n\right)}$	
$= \dfrac{5^{3n}}{5^n}$	
$= 5^{2n}$	

LS = RS

c.

LS	RS
$16^n + 16^n + 16^n + 16^n$	4^{2n+1}
$= 4 \times 16^n$	
$= 4^1 \times \left(4^2\right)^n$	
$= 4^{2n+1}$	

LS = RS

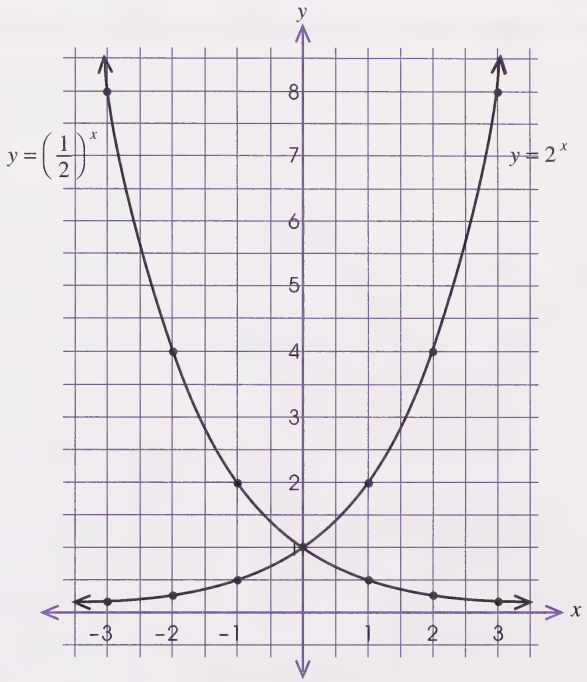
Section 1: Activity 2

1. $y = 2^x$

x	y
-3	$\frac{1}{8}$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

$y = \left(\frac{1}{2}\right)^x$

x	y
-3	8
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$



2. a. There are no x-intercepts for either function.

b. Both functions have a y-intercept of 1.

c. The domain for each function is the set of all real numbers R.

d. The range of each function is the set of all real numbers greater than zero. $\{y \mid y > 0,\ y \in R\}$

e. The intersection point of the graphs is $(0,1)$.

f. Yes, because $m^0 = 1$ for **all** replacements of m (except 0).

3. **a.** True **b.** False **c.** True
d. True **e.** True **f.** True

4. **a.** Use the graph $y = 2^x$. Locate 15 on the y-axis; then look where a horizontal line from 15 intersects the curve. Estimate the x-value. The x-value is approximately 3.9.

b. Use the graph $y = 10^x$. Locate 1.5 on the x-axis; then draw a vertical line from 1.5 until it intersects the graph. From the point on the curve draw a horizontal line until it meets the y-axis. This y-value is approximately 32.

c. Use the graph $y = 3^x$. From 40 on the y-axis, a horizontal line intersects the curve at an x-value of approximately 3.4.

d.
$$9^x = 30$$
$$\left(3^2\right)^x = 30$$
$$3^{2x} = 30$$

Now use the graph of $y = 3^x$ to estimate the value of x when y is 30. Locate 30 on the y-axis. A horizontal line from 30 intersects the curve at an x-value of approximately 3.2.

Thus, $3^{3.2} \doteq 30$.

Since $3^{3.2} \doteq 30$ and $3^{2x} = 30$, $2x \doteq 3.2$.

Thus, $x \doteq 1.6$.

e.
$$8^{0.8} = y$$
$$\left(2^3\right)^{0.8} = y$$
$$2^{2.4} = y$$

Now use the graph for $y = 2^x$. Locate 2.4 on the x-axis. A vertical line from 2.4 intersects the curve at a y-value of approximately 5. Thus, $y \doteq 5$.

Section 1: Activity 3

1. **a.**
$$7^{x+1} = 343$$
$$7^{x+1} = 7^3$$
$$\therefore\ x + 1 = 3$$
$$x = 2$$

b.
$$3^{x+2} = 81$$
$$3^{x+2} = 3^4$$
$$\therefore\ x + 2 = 4$$
$$x = 2$$

c.
$$2^{3x} = \frac{1}{2}$$
$$2^{3x} = 2^{-1}$$
$$\therefore\ 3x = -1$$
$$x = -\frac{1}{3}$$

d.
$$5^{2x} = \frac{1}{5}$$
$$5^{2x} = 5^{-1}$$
$$\therefore\ 2x = -1$$
$$x = -\frac{1}{2}$$

e. $4^{2x} = \left(8^{x}\right)^{x-2}$

$\left(2^{2}\right)^{2x} = 8^{x^{2}-2x}$

$2^{4x} = \left(2^{3}\right)^{x^{2}-2x}$

$2^{4x} = 2^{3x^{2}-6x}$

$\therefore\ 4x = 3x^{2} - 6x$

$0 = 3x^{2} - 10x$

$0 = x(3x-10)$

$x = 0$ or $3x - 10 = 0$

$x = \frac{10}{3}$

f. $9^{3x} = \left(27^{x}\right)^{x-3}$

$\left(3^{2}\right)^{3x} = 27^{x^{2}-3x}$

$3^{6x} = \left(3^{3}\right)^{x^{2}-3x}$

$3^{6x} = 3^{3x^{2}-9x}$

$\therefore\ 6x = 3x^{2} - 9x$

$0 = 3x^{2} - 15x$

$0 = 3x(x-5)$

$x = 0$ or $x - 5 = 0$

$x = 5$

2. a. $2\left(5^{2x-9}\right) = 250$

$5^{2x-9} = 125$

$5^{2x-9} = 5^{3}$

$\therefore\ 2x - 9 = 3$

$2x = 12$

$x = 6$

b. $3\left(7^{2x-5}\right) = 147$

$7^{2x-5} = 49$

$7^{2x-5} = 7^{2}$

$\therefore\ 2x - 5 = 2$

$2x = 7$

$x = 3.5$

c. $3^{x(x-4)} = 243$

$3^{x^{2}-4x} = 3^{5}$

$\therefore\ x^{2} - 4x = 5$

$x^{2} - 4x - 5 = 0$

$(x-5)(x+1) = 0$

$x - 5 = 0$ or $x + 1 = 0$

$x = 5$ $\quad$ $x = -1$

d. $5^{x(x+2)} = 125$

$5^{x^{2}+2x} = 5^{3}$

$\therefore\ x^{2} + 2x = 3$

$x^{2} + 2x - 3 = 0$

$(x-1)(x+3) = 0$

$x - 1 = 0$ or $x + 3 = 0$

$x = 1$ $\quad$ $x = -3$

e. $\left(\frac{1}{9}\right)^{1.5x} = \left(\frac{1}{27^{x}}\right)^{x-1}$

$\left(9^{-1}\right)^{1.5x} = \left(27^{-x}\right)^{x-1}$

$\left(3^{-2}\right)^{1.5x} = \left(3^{-3x}\right)^{x-1}$

$3^{-3x} = 3^{-3x^{2}+3x}$

$\therefore\ -3x = -3x^{2} + 3x$

$0 = -3x^{2} + 6x$

$0 = -3x(x-2)$

$-3x = 0$ or $x - 2 = 0$

$x = 0$ $\quad$ $x = 2$

f. $\left(\frac{1}{16}\right)^{3x} = \left(\frac{1}{64^{x}}\right)^{x-2}$

$\left(16^{-1}\right)^{3x} = \left(64^{-x}\right)^{x-2}$

$\left(4^{-2}\right)^{3x} = \left(4^{-3x}\right)^{x-2}$

$4^{-6x} = 4^{-3x^{2}+6x}$

$\therefore\ -6x = -3x^{2} + 6x$

$3x^{2} - 12x = 0$

$3x(x-4) = 0$

$x = 0$ or $x - 4 = 0$

$x = 4$

Section 1: Activity 4

1. $A = A_0\left(2^{\frac{t}{p}}\right)$

$1920 = 120\left(2^{\frac{3}{p}}\right)$

$16 = 2^{\frac{3}{p}}$

$2^4 = 2^{\frac{3}{p}}$

$\therefore\ 4 = \frac{3}{p}$

$p = \frac{3}{4}$ or 0.75

The doubling period is 0.75 h.

2. a. $A = A_0\left(3^{\frac{t}{p}}\right)$

$1620 = A_0\left(3^{\frac{8}{2}}\right)$

$1620 = A_0\left(3^4\right)$

$20 = A_0$

The initial number of bacteria is 20.

b. $A = A_0\left(3^{\frac{t}{p}}\right)$

$= 20\left(3^{\frac{16}{2}}\right)$

$= 20\left(3^8\right)$

$= 20 \times 6561$

$= 131\,220$

The number of bacteria after 16 h is 131 220.

3. a. Let $a_0 = 1$.

$a = a_0\left(2^{-\frac{t}{h}}\right)$

$a = 1\left(2^{-\frac{8}{24}}\right)$

$= \frac{1}{2^{\frac{1}{3}}}$ or $\frac{1}{\sqrt[3]{2}}$

The amount remaining is $\frac{1}{\sqrt[3]{2}}$ of the original amount.

b. $a = 1\left(2^{-\frac{72}{24}}\right)$

$= 1\left(2^{-3}\right)$

$= \frac{1}{2^3}$

$= \frac{1}{8}$

The amount remaining is $\frac{1}{8}$ of the original amount.

4. $a = a_0 \times 2^{-\frac{t}{h}}$

$1 = 4 \times \left(\frac{1}{2}\right)^{\frac{t}{4.6}}$

$\frac{1}{4} = \left(\frac{1}{2}\right)^{\frac{t}{4.6}}$

$\left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^{\frac{t}{4.6}}$

$\therefore\ 2 = \frac{t}{4.6}$

$t = 9.2$ d

It takes 9.2 d for the substance to decay to 1 g.

Section 1: Follow-up Activities

Extra Help

1. a. The **Product** Law is represented by $a^m \bullet a^n = \boldsymbol{a^{m+n}}$.

b. The **Power of a Power** Law is represented by $\left(a^m\right)^n = \boldsymbol{a^{mn}}$.

c. The **Power of a Product** Law is represented by $(ab)^m = \boldsymbol{a^m b^m}$.

d. The **Quotient** Law is represented by $a^m \div a^n = \boldsymbol{a^{m-n}}$.

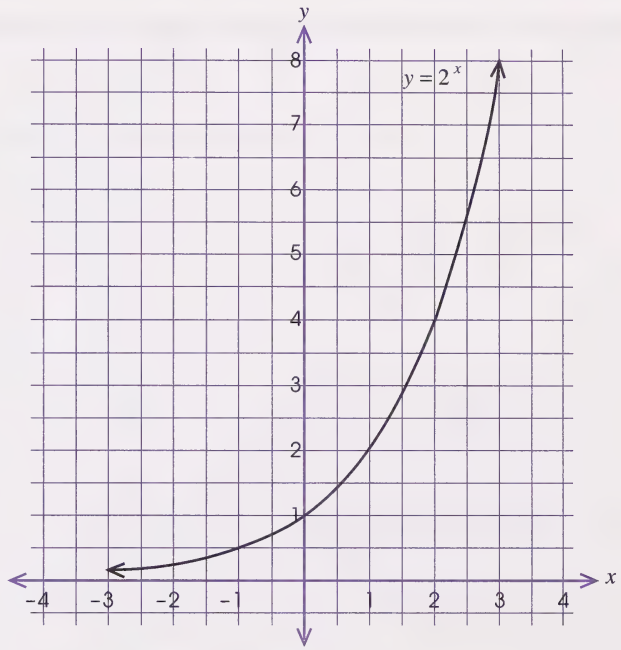
e. The **Power of a Quotient** Law is represented by $\left(\frac{a}{b}\right)^m = \frac{\boldsymbol{a^m}}{\boldsymbol{b^m}}$.

2. a. $a^0 = \mathbf{1}$

b. $a^{-n} = \dfrac{\mathbf{1}}{\boldsymbol{a^n}}$ or $\dfrac{1}{a^{-n}} = \boldsymbol{a^n}$

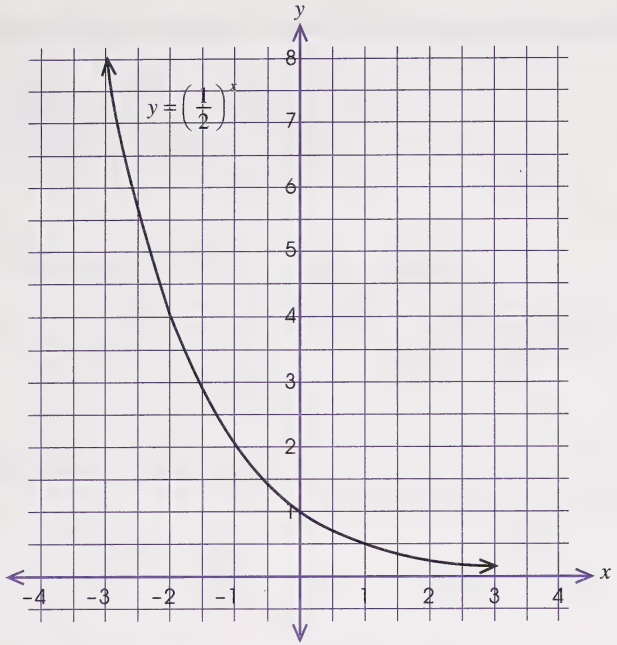
c. $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$ or $\sqrt[n]{\boldsymbol{a^m}}$

3.



As y increases, x increases to the right.

4.



As y increases, x decreases to the right.

5. a. The doubling period is the period of **time** required for a given quantity to double in size.

b. Half-life is the period of time during which a given amount of radioactive material decays until **one-half** of it is left.

6. $A = A_0\, 3^{\frac{t}{P}}$

$A = 500\left(3^{\frac{16}{4}}\right)$

$= 40\,500$

There is 40 500 bacteria after 16 h.

7. $a = a_0\left(2^{-\frac{t}{h}}\right)$

$= 4\left(2^{-\frac{60}{10}}\right)$

$= 4\left(2^{-6}\right)$

$= 2^2\left(2^{-6}\right)$

$= 2^{-4}$

$= \frac{1}{16}$ g

After 60 d, there is $\frac{1}{16}$ g of substance remaining.

Enrichment

1. a. $(18)^{\frac{1}{5}} \doteq \left(3^{2.6}\right)^{\frac{1}{5}}$

$\doteq 3^{0.52}$

$\doteq 1.7$

b. $\sqrt{6^3} = 6^{\frac{3}{2}}$

$\doteq \left(3^{1.6}\right)^{\frac{3}{2}}$

$\doteq 3^{2.4}$

$\doteq 14.3$

c. $(10)^{\frac{1}{3}} \times \sqrt[3]{14^2} \doteq \left(3^{2.1}\right)^{\frac{1}{3}} \times \left(3^{2.4}\right)^{\frac{2}{3}}$

$\doteq 3^{0.7} \times 3^{1.6}$

$\doteq 3^{0.7+1.6}$

$\doteq 3^{2.3}$

$\doteq 12.5$

2. $3^{x+1} = 73$

$3^{x+1} \doteq 9 \times 8.1 \qquad (73 \div 9 \doteq 8.1)$

$3^{x+1} \doteq 3^2 \times 3^{1.9} \qquad$ (From the graph, $8.1 \doteq 3^{1.9}$.)

$3^{x+1} \doteq 3^{3.9}$

$\therefore x+1 \doteq 3.9$

$x \doteq 2.9$

Section 2: Activity 1

1. a. $x = 5^y$ **b.** $y = 3^x$

c. $s = r^q$ **d.** $g = d^c$

2. a. $d = a^y$ **b.** $y = \log_a d$

Section 2: Activity 2

1. Domain: $\{x \mid x > 0,\ x \in R\}$

Range: $\{y \mid y \in R\}$

2. The graph $y = \log_b x$ is equivalent to $x = b^y$. Since $y = b^x$ is the inverse of $x = b^y$, the domain of $y = b^x$ is the range of $x = b^y$, and the range of $y = b^x$ is the domain of $x = b^y$. Thus, Q has the coordinates $\left(b^a, a\right)$.

3. a. True

b. True

c. True

d. False, $x = 1$.

e. True

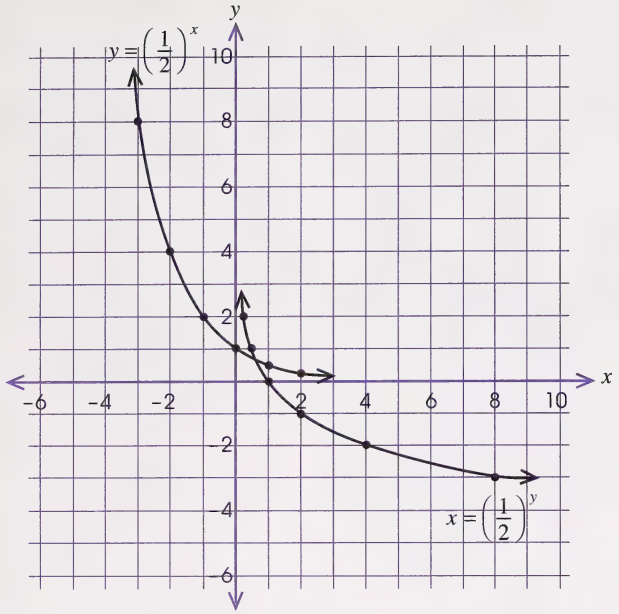
f. True, $\log_5 x = y$ is equivalent to $x = 5^y$. The inverse of $x = 5^y$ is $y = 5^x$. The logarithmic form of $y = 5^x$ is $\log_5 y = x$.

4. $y = \left(\frac{1}{2}\right)^x$

x	-3	-2	-1	0	1	2
y	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

$x = \left(\frac{1}{2}\right)^y$

x	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$
y	-3	-2	-1	0	1	2



5. **a.** $\log_{10} 4 \doteq 0.6$

b. $\log_{10} 8 \doteq 0.9$

c. $\log_{10} 10 \doteq 1.0$

d. $x \doteq 1.5$

e. $x \doteq 5.5$

Section 2: Activity 3

1. **a.** $\log_3 81 = 4$ **b.** $\log_2 16 = 4$

c. $\log_2 32 = 5$ **d.** $\log_4 64 = 3$

e. $\log_{\frac{1}{4}} \frac{1}{16} = 2$ **f.** $\log_{\frac{1}{3}} \frac{1}{9} = 2$

g. $\log_2 \frac{1}{8} = -3$ **h.** $\log_3 \frac{1}{9} = -2$

i. $\log_4 1 = 0$ **j.** $\log_6 1 = 0$

k. $\log_a c = b$ **l.** $\log_b d = a$

2. **a.** $2^4 = 16$ **b.** $3^4 = 81$

c. $4^3 = 64$ **d.** $5^2 = 25$

e. $5^{-1} = \frac{1}{5}$ **f.** $4^{-1} = \frac{1}{4}$

g. $10^2 = 100$ **h.** $10^4 = 10\,000$

i. $a^c = b$ **j.** $c^e = d$

k. $10^{-2} = 0.01$ **l.** $10^{-1} = 0.1$

Section 2: Activity 4

1. **a.**

$3 = 9^x$

$3 = 3^{2x}$

$2x = 1$

$x = \frac{1}{2}$

b.

$16 = 8^y$

$2^4 = 2^{3y}$

$3y = 4$

$y = \frac{4}{3}$

c. $0.001 = 10^{m}$

$$\frac{1}{1000} = 10^{m}$$

$$10^{-3} = 10^{m}$$

$$m = -3$$

d. $27 = \left(\sqrt{3}\right)^{x}$

$$3^{3} = \left(\sqrt{3}\right)^{x}$$

$$\left[\left(\sqrt{3}\right)^{2}\right]^{3} = \left(\sqrt{3}\right)^{x}$$

$$\left(\sqrt{3}\right)^{6} = \left(\sqrt{3}\right)^{x}$$

$$x = 6$$

e. $0.1 = 10^{y}$

$$\frac{1}{10} = 10^{y}$$

$$10^{-1} = 10^{y}$$

$$y = -1$$

f. $16 = \left(\sqrt{2}\right)^{y}$

$$2^{4} = \left(\sqrt{2}\right)^{y}$$

$$\left[\left(\sqrt{2}\right)^{2}\right]^{4} = \left(\sqrt{2}\right)^{y}$$

$$\left(\sqrt{2}\right)^{8} = \left(\sqrt{2}\right)^{y}$$

$$y = 8$$

g. $\sqrt[4]{81} = 9^{x}$

$$\left(3^{4}\right)^{\frac{1}{4}} = 3^{2x}$$

$$3 = 3^{2x}$$

$$2x = 1$$

$$x = \frac{1}{2}$$

h. $\sqrt[3]{25} = 125^{m}$

$$\left(5^{2}\right)^{\frac{1}{3}} = 5^{3m}$$

$$5^{\frac{2}{3}} = 5^{3m}$$

$$3m = \frac{2}{3}$$

$$m = \frac{2}{9}$$

2. a. $\log_4 16 = \log_4 4^2$

$$= 2$$

b. $\log_3 729 = \log_3 3^6$

$$= 6$$

c. $\log_7 49 = \log_7 7^2$

$$= 2$$

d. $\log_2 \frac{1}{8} = \log_2 2^{-3}$

$$= -3$$

3. a. Let $x = 8^{\log_5 25}$.

$$\therefore x = 8^{\log_5 5^2}$$

$$= 8^2$$

$$= 64$$

b. Let $x = 2^{\log_2 32}$.

$$\therefore x = 2^{\log_2 2^5}$$

$$= 2^5$$

$$= 32$$

c. Let $x = \log_4\left(\log_4 16\right)$.

$$\therefore x = \log_4\left(\log_4 4^2\right)$$

$$x = \log_4 2$$

$$2 = 4^{x}$$

$$2 = 2^{2x}$$

$$2x = 1$$

$$x = \frac{1}{2}$$

d. Let $x = \log_{\frac{1}{3}}\left(\log_4 4^9\right)$.

$$\therefore x = \log_{\frac{1}{3}} 9$$

$$9 = \left(\frac{1}{3}\right)^{x}$$

$$3^2 = 3^{-x}$$

$$-x = 2$$

$$x = -2$$

4. **a.** 1.18 **b.** 1.30
c. 18 620.87 **d.** 4897.79
e. 3.64 **f.** 3.97
g. 0.77 **h.** 0.71
i. 7.08 **j.** 2.34
k. 37.15 **l.** 107.15

5. **a.** $x = 4^3$
$= 64$

b. $x = 3^2$
$= 9$

6. **a.** $x = 9^{\frac{1}{2}}$
$= 3$

b. $x = 16^{\frac{1}{2}}$
$= 4$

7. **a.** $x^3 = 8$
$x^3 = 2^3$
$x = 2$

b. $x^2 = 64$
$x = 8$

8. **a.** $x = 1.4$

b. $x = 2.8$

9. **a.** $x^3 = 125$
$x^3 = 5^3$
$x = 5$

b. $x^3 = 64$
$x^3 = 4^3$
$x = 4$

10. **a.** $9^{-\frac{3}{2}} = x$
$\frac{1}{9^{\frac{3}{2}}} = x$
$\frac{1}{\left(\sqrt{9}\right)^3} = x$
$\frac{1}{27} = x$

b. $16^{-\frac{3}{4}} = x$
$\frac{1}{16^{\frac{3}{4}}} = x$
$\frac{1}{\left(\sqrt[4]{16}\right)^3} = x$
$\frac{1}{8} = x$

11. **a.** $x^{-\frac{2}{3}} = \frac{1}{16}$
$\frac{1}{x^{\frac{2}{3}}} = \frac{1}{16}$
$x^{\frac{2}{3}} = 16$
$\left[(x)^{\frac{2}{3}}\right]^{\frac{3}{2}} = 16^{\frac{3}{2}}$
$x^{\frac{6}{6}} = \left(\sqrt[2]{16}\right)^3$
$x^1 = 4^3$
$x = 64$

b. $x^{-\frac{5}{2}} = \frac{1}{32}$
$\frac{1}{x^{\frac{5}{2}}} = \frac{1}{32}$
$x^{\frac{5}{2}} = 32$
$\left[x^{\frac{5}{2}}\right]^{\frac{2}{5}} = 32^{\frac{2}{5}}$
$x^1 = \left(\sqrt[5]{32}\right)^2$
$x = 4$

12. a. Method 1

$$\begin{aligned} x &= \log_4 64^{-1} \\ &= \log_4 \left(4^3\right)^{-1} \\ &= \log_4 4^{-3} \\ &= -3 \end{aligned}$$

Method 2

$$\begin{aligned} 4^x &= 64^{-1} \\ &= \frac{1}{64} \\ &= \frac{1}{4^3} \\ 4^x &= 4^{-3} \end{aligned}$$

Thus, $x = -3$.

b.
$$\begin{aligned} x &= \log_5 25^{-\frac{3}{5}} \\ &= \log_5 \left(5^2\right)^{-\frac{3}{5}} \\ &= \log_5 5^{-\frac{6}{5}} \\ &= -\frac{6}{5} \end{aligned}$$

13. a.
$$\begin{aligned} x &= \log_3 \sqrt[5]{3} \\ &= \log_3 3^{\frac{1}{5}} \\ &= \frac{1}{5} \end{aligned}$$

b.
$$\begin{aligned} x &= \log_5 25^{\frac{1}{5}} \\ &= \log_5 \left(5^2\right)^{\frac{1}{5}} \\ &= \log_5 5^{\frac{2}{5}} \\ &= \frac{2}{5} \end{aligned}$$

14. a.
$$\begin{aligned} x &= 3^{\log_3 \frac{1}{9}} \\ &= 3^{-2} \qquad \left(\text{since } \log_3 \tfrac{1}{9} = -2.\right) \\ &= \frac{1}{3^2} \\ &= \frac{1}{9} \end{aligned}$$

b.
$$\begin{aligned} x &= 5^{\log_5 5^2} \\ &= 5^2 \\ &= 25 \end{aligned}$$

15.
$$\log_4 A = \log_4 B + \log_4 C$$
$$\log_4 A = \log_4 16 + \log_4 256$$
$$\log_4 A = \log_4 4^2 + \log_4 4^4$$
$$\log_4 A = 2 + 4$$
$$\log_4 A = 6$$
$$A = 4^6$$

Section 2: Follow-up Activities

Extra Help

1. a. The **inverse** of the exponential function $y = b^x$, where $b > 0$ and $b \neq 1$, is obtained by interchanging the domain and **range** to give $\boldsymbol{x = b^y}$.

b. The expression $y = \log_b x$ is read as "y is the logarithm of x to the base b." The exponential form $x = b^y$ is equivalent to $y = \log_b x$.

c. In order to draw the graph of $y = \log_3 x$ change this logarithmic form to the exponential form $x = 3^y$. To prepare a table of values, choose values for y, then calculate corresponding x-values. To find the value of $y = \log_3 12$ from the plotted curve, locate 12 on the x-axis; then draw a **vertical** line until it intersects the curve. Then draw a **horizontal** line until it meets the y-axis.

d. The statement $\log 1000 = 3$ means that the base is **10**, and this is called the **common** logarithm.

e. The statement $3^5 = 3^x$, means that $x = 5$; because when bases are equal, the **exponents** are equal and vice versa.

f. When solving expressions of the form $x = \log_a a^m$, then x has the value of m.

g. When solving expressions of the form $x = a^{\log_a d}$, then x has the value of d.

h. The expression $\log_a a$, where $a > 0$ and $a \neq 1$, has the value **1**.

2. a. $x = a^y$

b. $x = a^y \Leftrightarrow y = \log_a x$

3. a. $\log_2 14 \doteq 3.8$

b. $x \doteq 3$

4. a. $y = \log_5 x$

b. $\log_4 64 = 3$

5. a. $a^d = h$

b. $7^{-2} = \frac{1}{49}$

c. $10^1 = 10$

d. $10^{-3} = 0.001$

6. a.
$$\begin{aligned} x &= 4^2 \\ &= 16 \end{aligned}$$

b.
$$\begin{aligned} 2^x &= 32 \\ 2^x &= 2^5 \\ x &= 5 \end{aligned}$$

c.
$$\begin{aligned} x^3 &= 216 \\ x^3 &= 6^3 \\ x &= 6 \end{aligned}$$

d.
$$\begin{aligned} 8^{-\frac{2}{3}} &= x \\ \frac{1}{8^{\frac{2}{3}}} &= x \\ \frac{1}{\left(\sqrt[3]{8}\right)^2} &= x \\ \frac{1}{4} &= x \end{aligned}$$

e.
$$\begin{aligned} x^{-\frac{3}{2}} &= \frac{1}{64} \\ \frac{1}{x^{\frac{3}{2}}} &= \frac{1}{64} \\ x^{\frac{3}{2}} &= 64 \\ \left(x^{\frac{3}{2}}\right)^{\frac{2}{3}} &= (64)^{\frac{2}{3}} \\ x &= \left(\sqrt[3]{64}\right)^2 \\ &= 4^2 \\ &= 16 \end{aligned}$$

f. $x = 2.5$ (If $x = \log_a a^m$, then $x = m$.)

g. $\log_5 5^{\frac{1}{4}} = x$

$$\frac{1}{4} = x$$

h. $6^{\log_6 \frac{1}{32}} = x$ (If $x = a^{\log_a d}$, then $x = d$.)

$$\frac{1}{32} = x$$

7. a. $\log_5 \left(\log_7 7^{25} \right) = x$

$$\log_5 25 = x$$

$$\log_5 5^2 = x$$

$$2 = x$$

b. $\log_4 \left(3^{\log_3 2} \right) = x$

$$\log_4 2 = x$$

$$4^x = 2$$

$$\left(2^2 \right)^x = 2$$

$$2^{2x} = 2^1$$

$$2x = 1$$

$$x = \frac{1}{2}$$

8. a. 1.51 **b.** 2.39
c. 12.02 **d.** 5.37

Enrichment

1. a. For b greater than one and the larger the base, the graph **increases** more rapidly from left to right.

b. For $0 < b < 1$ and the smaller the base, the graph **decreases** more rapidly from left to right.

c. All graphs pass through the point $(\mathbf{0, 1})$.

d. Since $1^x = 1$ for all x, the graph of $y = 1^x$ is the **horizontal** line passing through the point $(\mathbf{0, 1})$.

2. a. For b greater than one and the larger the base, the graph increases more **slowly** from left to right.

b. For $0 < b < 1$ and the larger the base, the graph decreases more **rapidly** from left to right.

c. All graphs pass through the point $(\mathbf{1, 0})$.

3. a. $\log_3 \frac{1}{3} = -1$ and $\log_3 6561 = 8$

Add the logs.

$-1 + 8 = 7$

Thus, $\frac{1}{3} \times 6561 = 2187$. (2187 is opposite 7.)

b. $\log_3 81 = 4$ and $\log_3 \frac{1}{9} = -2$

Subtract the logs.

$4 - (-2) = 6$

Thus, $81 \div \frac{1}{9} = 729$. (729 is opposite 6.)

c. $\log_3 9 = 2$

Thus, $\log_3 (9 \times 9 \times 9 \times 9) = 2 + 2 + 2 + 2$

$= 8$

Therefore, $9^4 = 6561$. (8 is opposite 6561.)

d. $\log_3 19\,683 = 9$ (from table of values)

Thus, $\sqrt[3]{19\,683} = 19\,683^{\frac{1}{3}}$

$= \left(3^9\right)^{\frac{1}{3}}$

$= 3^3$

$= 27$

Section 3: Activity 1

1. a. $\log_5 6 + \log_5 12$ **b.** $\log_3 4 + \log_3 16$

c. $\log_7 4 + \log_7 11$ **d.** $\log 3 + \log 9$

2. a. $\log_2 (12 \times 6) = \log_2 72$

b. $\log_4 (7 \times 11) = \log_4 77$

c. $\log_b (ac)$

d. $\log_3 (2 \times 5 \times 8) = \log_3 (80)$

e. $\log_x \left(a^2 b \times a^3 b\right) = \log_x \left(a^5 b^2\right)$

f. $\log (7.3 \times 18.4) = \log (134.32)$

3. a. $\log_7 (49) = \log_7 7^2$

$= 2$

b. $\log_2 64 = \log_2 2^6$

$= 6$

c. $\log_5 625 = \log_5 5^4$

$= 4$

d. $\log_3 2187 = \log_3 3^7$

$= 7$

4. a. $\log_7 5.2 - \log_7 1.7$ **b.** $\log_5 8.9 - \log_5 7.6$

c. $\log_2 39 - \log_2 14$ **d.** $\log_m p - \log_m q$

5. a. $\log_{27} 35.2 + \log_{27} 1.5 - \log_{27} 17.1$

b. $\log_{14} 24 + \log_{14} 2.7 - \log_{14} 18.5$

c. $\log_b x + \log_b y - \log_b z$

d. $\log_c x - \log_c y - \log_c z$

6. a. $\log_3 \frac{27.2}{8.1}$ **b.** $\log_5 \frac{37}{146}$

7. a. $\log_x \frac{(9.3)(0.6)}{19}$ **b.** $\log \frac{(5.8)(40)}{60}$

c. $\log_4 \frac{(15)(7.4)}{18}$

8. a. $\log \frac{15.2}{7.6} = \log 2$

$= 0.3$

b. $\log \frac{26.1}{8.7} = \log 3$

$= 0.5$

c. $\log\dfrac{(15.1)(7.8)}{26.3} = \log 4.478$

$= 0.7$

d. $\log\dfrac{(8.9)(5.3)}{17.6} = \log 2.6801$

$= 0.4$

9. a. $\log_2 \dfrac{84}{5.25} = \log_2 16$

$= \log_2 2^4$

$= 4$

b. $\log_3 \dfrac{58.5}{6.5} = \log_3 9$

$= \log_3 3^2$

$= 2$

c. $\log_6 \dfrac{(12)(9)}{3} = \log_6 36$

$= \log_6 6^2$

$= 2$

d. $\log_4 \dfrac{(20)(16)}{5} = \log_4 64$

$= \log_4 4^3$

$= 3$

10. a. $2\log_2 3$ **b.** $3\log_4 5$

c. $\dfrac{1}{3}\log_3 5$ **d.** $\dfrac{2}{3}\log_6 4$

e. $-\log_c x$ **f.** $-2\log_a y$

g. $\dfrac{1}{2}\log 5$ **h.** $\dfrac{4}{3}\log 5$

i. $-2\log_b x$ **j.** $3\log_a y$

11. a. $\log_3 5^2$ or $\log_3 25$ **b.** $\log_4 7^3$ or $\log_4 343$

c. $\log_9 7^{\frac{2}{3}}$ or $\log_9 \sqrt[3]{49}$ **d.** $\log_8 23^{\frac{3}{5}}$ or $\log_8 \sqrt[5]{23^3}$

e. $\log_b x^n$ **f.** $\log_c d^a$

g. $\log_p y^{\frac{1}{x}}$ or $\log_p \sqrt[x]{y}$ **h.** $\log_b x^{\frac{1}{a}}$ or $\log_b \sqrt[a]{x}$

12. a. $\log\dfrac{MN^2}{P}$

b. $\log\dfrac{A^2B}{C^3}$

c. $\log_3 \dfrac{xy}{ab}$

d. $\log_2 \dfrac{ab}{cd}$

e. $\log_b \left[2x\left(\dfrac{x}{y}\right)\right] = \log_b \dfrac{2x^2}{y}$

f. $\log_3 \left[4a\left(\dfrac{a}{c}\right)\right] = \log_3 \dfrac{4a^2}{c}$

g. $\log_a \frac{\frac{a}{x}}{ax} = \log_a \frac{1}{x^2}$

h. $\log_b \frac{\frac{c}{x^2}}{cx^3} = \log_b \frac{1}{x^5}$

i. $\log_5 5^n = n\log_5 5$

If $\log_5 5 = 1$, then $n\log_5 5 = n(1) = n$.

Therefore, $\log_5 5^n = n$.

13. a. $\log_3 4 = \log_3 (2\times 2)$
$= \log_3 2 + \log_3 2$
$\doteq 0.63 + 0.63$
$\doteq 1.26$

b. $\log_3 25 = \log_3 (5\times 5)$
$= \log_3 5 + \log_3 5$
$\doteq 1.465 + 1.465$
$\doteq 2.930$

c. $\log_3 10 = \log_3 (2\times 5)$
$= \log_3 2 + \log_3 5$
$\doteq 0.63 + 1.465$
$\doteq 2.095$

d. $\log_3 20 = \log_3 (2\times 2\times 5)$
$= \log_3 2 + \log_3 2 + \log_3 5$
$\doteq 0.63 + 0.63 + 1.465$
$\doteq 2.725$

e. $\log_3 \frac{5}{2} = \log_3 5 - \log_3 2$
$\doteq 1.465 - 0.63$
$\doteq 0.835$

f. $\log_3 \frac{2}{5} = \log_3 2 - \log_3 5$
$\doteq 0.63 - 1.465$
$\doteq -0.835$

g. $\log_3 0.25 = \log_3 \frac{1}{4}$
$= \log_3 1 - \log_3 (2\times 2)$
$= \log_3 1 - (\log_3 2 + \log_3 2)$
$\doteq 0 - (0.63 + 0.63)$
$\doteq -1.26$

h. $\log_3 0.125 = \log_3 \frac{1}{8}$

$= \log_3 1 - \log_3 (2 \times 2 \times 2)$

$= \log_3 1 - (\log_3 2 + \log_3 2 + \log_3 2)$

$\doteq 0 - (0.63 + 0.63 + 0.63)$

$\doteq -1.89$

Section 3: Activity 2

1. a. $\frac{1}{2}\log 5 \doteq \frac{1}{2}(0.6990)$

$\doteq 0.35$

b. $\frac{1}{3}\log 7 \doteq \frac{1}{3}(0.8451)$

$\doteq 0.28$

c. $\frac{1}{3}\log 15.2 \doteq \frac{1.1818}{3}$

$\doteq 0.39$

d. $\frac{1}{4}\log 5 = \frac{1}{4}(0.6990)$

$\doteq 0.17$

e. $\frac{2}{3}\log 5.2 + \frac{1}{5}\log 3.8 \doteq \frac{2}{3}(0.7160) + \frac{1}{5}(0.5798)$

$\doteq 0.4773 + 0.1160$

$\doteq 0.59$

f. $\frac{3}{2}\log 6 + \frac{3}{8}\log 4 \doteq \frac{3}{2}(0.7782) + \frac{3}{8}(0.6021)$

$\doteq 1.1672 + 0.2258$

$\doteq 1.39$

2. a. $\frac{1}{3}\log_3 9 = \frac{1}{3}(2)$

$= \frac{2}{3}$

b. $\log_4 64 + \frac{1}{5}\log_4 4 = 3 + \frac{1}{5}(1)$

$= 3\frac{1}{5}$

c. $\log_5 25 + \frac{1}{2}\log_5 125 + \log_3 81 + \frac{1}{3}\log_3 243$

$= 2 + \frac{1}{2}(3) + 4 + \frac{1}{3}(5)$

$= 9\frac{1}{6}$

3. a. $\frac{1}{4}\log_6 3 \doteq \frac{1}{4}(0.613)$

$\doteq 0.153$

b. $\frac{1}{4}\log_6 9 = \frac{1}{4}\log_6 (3\times 3)$

$= \frac{1}{4}\left(\log_6 3 + \log_6 3\right)$

$\doteq \frac{1}{4}(0.613 + 0.613)$

$\doteq \frac{1}{4}(1.226)$

$\doteq 0.307$

c. $\log_6 (36)(3) = \log_6 36 + \log_6 3$

$\doteq 2 + 0.613$

$\doteq 2.613$

4. a. $\log\frac{AB^3}{C^4}$

b. $\log\left(\frac{AB}{C}\right)^{\frac{1}{3}}$

c. $\log_b \frac{A^{\frac{1}{4}}B^5}{A^3} = \log_b A^{-\frac{11}{4}}B^5$

d. $\log_5 \frac{\sqrt[7]{AB^3}}{(CD)^2}$

e. $\log_2 \sqrt[3]{\frac{A^2}{B^5C^4}} = \log_2 \frac{A^{\frac{2}{3}}}{B^{\frac{5}{3}}C^{\frac{4}{3}}}$

f. $\log_b \left(\frac{c}{y^3}\right)^{\frac{1}{2}} - \log_b (cy)^{\frac{1}{2}} = \log_b c^{\frac{1}{2}}y^{-\frac{3}{2}} - \log_b c^{\frac{1}{2}}y^{\frac{1}{2}}$

$= \log_b \frac{c^{\frac{1}{2}}y^{-\frac{3}{2}}}{c^{\frac{1}{2}}y^{\frac{1}{2}}}$

$= \log_b \left(y^{-\frac{3}{2}}y^{-\frac{1}{2}}\right)$

$= \log_b y^{-2}$

Section 3: Follow-up Activities

Extra Help

1. The Product Law of Logarithms states that $\log_b (mn) = \mathbf{\log_b m + \log_b n}$.

2. The Product Law of Logarithms indicates that the log of a product equals the sum of the log of each **factor**.

3. The Quotient Law of Logarithms states that $\log_b \frac{m}{n} = \mathbf{\log_b m - \log_b n}$.

4. The Quotient Law of Logarithms indicates that the log of a quotient equals the **difference** of the log of the dividend and the log of the divisor.

5. The Power Law of Logarithms states that $\log_b m^n = \mathbf{n\log_b m}$.

6. $\log_2 5 + \log_2 4$

7. $\log_c \dfrac{20 \times 16}{7}$

8. $\log \dfrac{(18.1)(12.4)}{19.3} \doteq \log 11.629$

$\doteq 1.07$

9. $\log_5 24 = \log_5 (4 \times 6)$

$= \log_5 4 + \log_5 6$

$\doteq 0.861 + 1.113$

$\doteq 1.974$

10. $\log_5 \dfrac{10 \times 15}{6} = \log_5 25$

$= \log_5 5^2$

$= 2$

11. $\dfrac{1}{5}\log_7 3$

12. $\log_3 12^4$

13. $\dfrac{1}{3}\log 6 + \dfrac{2}{3}\log 5.3 \doteq \dfrac{1}{3}(0.778) + \dfrac{2}{3}(0.724)$

$\doteq 0.26 + 0.48$

$\doteq 0.74$

14. $\log_7 49 + \dfrac{1}{3}\log_7 7 = 2 + \dfrac{1}{3}(1)$

$= 2\dfrac{1}{3}$

15. $\dfrac{1}{4}\log_6 20 = \dfrac{1}{4}\log_6 (2 \times 2 \times 5)$

$= \dfrac{1}{4}\left(\log_6 2 + \log_6 2 + \log_6 5\right)$

$\doteq \dfrac{1}{4}(0.387 + 0.387 + 0.898)$

$\doteq 0.418$

16. $\log_5 \dfrac{\left(AB^2\right)^{\frac{1}{3}}}{C^7}$

Enrichment

1. a. $\log_7 A = \log_7 20 - \log_7 4$

$\log_7 A = \log_7 \frac{20}{4}$

$\log_7 A = \log_7 5$

$A = 5$

b. $3\log_2 A = 6\log_2 8$

$\log_2 A^3 = \log_2 8^6$

$A^3 = 8^6$

$A = 8^2$

$= 64$

c. $2\log_{24} A - \log_{24} 6 = 1$

$\log_{24} \frac{A^2}{6} = \log_{24} 24$

$\frac{A^2}{6} = 24$

$A^2 = 144$

$A = 12$ $(A = -12$ is inadmissible.)

2. $\log_5 \frac{3}{2} = \log_5 \frac{6}{4}$

$= \log_5 6 - \log_5 4$

$\doteq 1.113 - 0.861$

$\doteq 0.252$

Section 4: Activity 1

1. a. $4^x = 9$

$\log 4^x = \log 9$

$x \log 4 = \log 9$

$x = \frac{\log 9}{\log 4}$

$\doteq \frac{0.954\,24}{0.602\,06}$

$\doteq 1.584\,96$

b. $7^x = 15$

$\log 7^x = \log 15$

$x \log 7 = \log 15$

$x = \frac{\log 15}{\log 7}$

$\doteq 1.391\,66$

c. $6^{\frac{1}{4}x} = 23$

$\log 6^{\frac{1}{4}x} = \log 23$

$\frac{1}{4} x \log 6 = \log 23$

$x = \frac{4 \log 23}{\log 6}$

$\doteq \frac{4(1.361\,73)}{0.778\,15}$

$\doteq 6.999\,81$

d. $3^{\frac{1}{5}x} = 14$

$\log 3^{\frac{1}{5}x} = \log 14$

$\frac{1}{5} x \log 3 = \log 14$

$x = \frac{5 \log 14}{\log 3}$

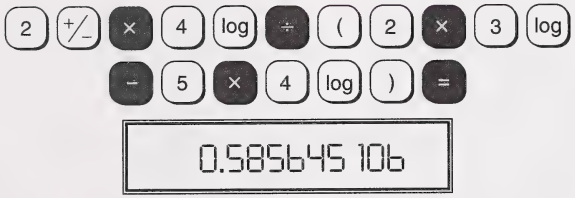
$\doteq \frac{5(1.146\,13)}{0.477\,12}$

$\doteq 12.010\,87$

e.

$$3^{2x} = 4^{5x-2}$$
$$\log 3^{2x} = \log 4^{5x-2}$$
$$2x\log 3 = (5x-2)\log 4$$
$$2x\log 3 = 5x\log 4 - 2\log 4$$
$$2x\log 3 - 5x\log 4 = -2\log 4$$
$$x[2\log 3 - 5\log 4] = -2\log 4$$
$$x = \frac{-2\log 4}{2\log 3 - 5\log 4}$$

Solve using a calculator.



Therefore, $x \doteq 0.585\,65$.

f.

$$5^{x} = 7^{3x-1}$$
$$\log 5^{x} = \log 7^{3x-1}$$
$$x\log 5 = (3x-1)\log 7$$
$$x\log 5 = 3x\log 7 - 1\log 7$$
$$x\log 5 - 3x\log 7 = -\log 7$$
$$x(\log 5 - 3\log 7) = -\log 7$$
$$x = \frac{-\log 7}{\log 5 - 3\log 7}$$
$$\doteq 0.460\,21$$

2. a. $\log_4 9 = \dfrac{\log 9}{\log 4}$

$\doteq 1.584\,96$

b. $\log_5 12 = \dfrac{\log 12}{\log 5}$

$\doteq 1.543\,96$

c. $\log_{\frac{1}{5}} 7 = \dfrac{\log 7}{\log\left(\frac{1}{5}\right)}$

$\doteq -1.209\,06$

d. $\log_{\frac{1}{4}} 8.6 = \dfrac{\log 8.6}{\log\left(\frac{1}{4}\right)}$

$\doteq -1.552\,17$

e. $\dfrac{1}{\log_3 8} = \dfrac{1}{\frac{\log 8}{\log 3}}$

$\doteq 0.528\,32$

f. $\dfrac{1}{\log_8 3} = \dfrac{1}{\frac{\log 3}{\log 8}}$

$\doteq 1.892\,79$

Section 4: Activity 2

1. a.

$$x^{\frac{2}{5}} = 32.6$$
$$\log x^{\frac{2}{5}} = \log 32.6$$
$$\frac{2}{5}\log x = \log 32.6$$
$$\log x = \frac{5}{2}\log 32.6$$
$$\log x \doteq 3.783\,044$$
$$10^{3.783\,044} \doteq x$$
$$x \doteq 6067.978$$

b.

$$x^{\frac{5}{8}} = 27.4$$
$$\log x^{\frac{5}{8}} = \log 27.4$$
$$\frac{5}{8}\log x = \log 27.4$$
$$\log x = \frac{8}{5}\log 27.4$$
$$\log x \doteq 2.300\,401$$
$$10^{2.300\,401} \doteq x$$
$$x \doteq 199.711$$

2. a.

$$\log_3 x + \log_3 6 = \log_3 5$$
$$\log_3 [(x)(6)] = \log_3 5$$
$$6x = 5$$
$$x = \frac{5}{6}$$

b.

$$\log_5 12 - \log_5 x = \log_5 8$$
$$\log_5 \frac{12}{x} = \log_5 8$$
$$\frac{12}{x} = 8$$
$$x = 1.5$$

3. a.

$$\log_5 2x - \log_5 7 = 1$$
$$\log_5 \frac{2x}{7} = 1$$
$$\frac{2x}{7} = 5^1$$
$$2x = 35$$
$$x = 17.5$$

b.

$$\log_3 3x + \log_3 6 = 1$$
$$\log_3 [(3x)(6)] = 1$$
$$(3x)(6) = 3^1$$
$$18x = 3$$
$$x = \frac{1}{6}$$

4. a.

$$\log_2 2x + \log_2 (x-1) = 2$$
$$\log_2 [(2x)(x-1)] = \log_2 2^2$$
$$(2x)(x-1) = 2^2$$
$$2x^2 - 2x = 4$$
$$2x^2 - 2x - 4 = 0$$
$$2\left(x^2 - x - 2\right) = 0$$
$$2(x+1)(x-2) = 0$$

$x+1=0$ or $x-2=0$

$x=-1$ $\qquad$ $x=2$

Since the log of a negative number does not exist, then the only possible answer is $x = 2$.

b. $\log_4 2x - \log_4 (2x-1) = 2$

$$\log_4 \frac{2x}{2x-1} = \log_4 4^2$$

$$\frac{2x}{2x-1} = 4^2$$

$$2x = 16(2x-1)$$

$$16 = 30x$$

$$x = \frac{8}{15}$$

5. a. $\log 6 - \log x = \log 3x + \log 2x$

$$\log \frac{6}{x} = \log[(3x)(2x)]$$

$$\log \frac{6}{x} = \log 6x^2$$

$$\frac{6}{x} = 6x^2$$

$$6 = 6x^3$$

$$x = 1$$

b. $\log 8x - \log 3x = \log 4 + \log x$

$$\log \frac{8x}{3x} = \log[(4)(x)]$$

$$\frac{8x}{3x} = 4x$$

$$8x = 12x^2$$

$$12x^2 - 8x = 0$$

$$4x(3x-2) = 0$$

$4x = 0$ or $3x - 2 = 0$

$x = 0$ $\qquad$ $x = \frac{2}{3}$

Since $x = 0$ will not satisfy the equation, then the only possible answer is $x = \frac{2}{3}$.

6. a. $3\log x = \log 8$

$$x^3 = 8$$

$$x = 2$$

b. $4\log x = \log \frac{1}{16}$

$$x^4 = \frac{1}{16}$$

$$x = \frac{1}{2}$$

Section 4: Activity 3

1.

$$G = G_0\left(2^{cy}\right)$$

$$8.03\times10^7 = 7.86\times10^7\left(2^{c(10)}\right)$$

$$\frac{8.03\times10^7}{7.86\times10^7} = 2^{10c}$$

$$\log\left(\frac{8.03\times10^7}{7.86\times10^7}\right) = 10c\log 2$$

$$c = \frac{\log\left(\frac{8.03\times10^7}{7.86\times10^7}\right)}{10\log 2}$$

$$\doteq \frac{0.009\,293}{3.010\,300}$$

$$\doteq 0.003\,087$$

For the year 2000, $G_0 = 7.86\times10^7$, $c = 0.003\,087$, and $t = 30$.

$$G \doteq 7.86\times10^7\left(2^{(0.003\,087)(30)}\right)$$

$$\doteq 8.38\times10^7$$

The population will be approximately 8.38×10^7.

2.

$$a = a_0\left(2^{-\frac{t}{h}}\right)$$

$$28 = a_0\left(2^{-\frac{62}{87}}\right)$$

$$\log 28 = \log a_0\left(2^{-\frac{62}{87}}\right)$$

$$\log 28 = \log a_0 + \log 2^{-\frac{62}{87}}$$

$$\log 28 = \log a_0 + \left(-\frac{62}{87}\log 2\right)$$

$$\log a_0 = \log 28 + \frac{62}{87}\log 2$$

$$\log a_0 \doteq 1.447\,158\,031 + 0.214\,527$$

$$\log a_0 \doteq 1.661\,685$$

$$a_0 \doteq 45.9$$

The original amount of the substance is about 46 g.

3. a.

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

$$= 1200\left(1+\frac{0.09}{4}\right)^{4(6)}$$

$$= 1200(1+0.0225)^{24}$$

$$= 1200(1.0225)^{24}$$

$$\doteq 2046.92$$

or

$$\log A = \log 1200 + 24\log 1.0225$$

$$A \doteq 3.0792 + 0.2319$$

$$\doteq 3.3111$$

$$\doteq 2046.92$$

The amount would be approximately \$2046.92.

b. $A = P\left(1+\frac{r}{n}\right)^{nt}$

$= 800\left(1+\frac{0.10}{2}\right)^{2(4)}$

$= 800(1+0.05)^{8}$

$= 1181.96$

The amount attained would be about \$1181.96.

4. a. $A = P\left(1+\frac{r}{n}\right)^{nt}$

$1600 = 800\left(1+\frac{0.08}{4}\right)^{4t}$

$\frac{1600}{800} = (1.02)^{4t}$

$\log 2 = \log(1.02)^{4t}$

$\log 2 = 4t\log(1.02)$

$t = \frac{\log 2}{4\log(1.02)}$

$\doteq \frac{0.301\,030}{4(0.008\,600)}$

$\doteq 8.75$

It will take about 8.75 years to double.

b. $A = P\left(1+\frac{r}{n}\right)^{nt}$

$2100 = 700\left(1+\frac{0.10}{2}\right)^{2t}$

$\frac{2100}{700} = (1.05)^{2t}$

$\log 3 = \log(1.05)^{2t}$

$\log 3 = 2t\log(1.05)$

$t = \frac{\log 3}{2\log(1.05)}$

$\doteq \frac{0.477\,12}{2(0.021\,189)}$

$\doteq 11.26$

Since it is compounded semi-annually, it will take about 11.5 years.

5. a. $\text{pH} = \log_{10}\frac{1}{\text{H}_c}$

$7.5 = \log_{10}\frac{1}{\text{H}_c}$

$7.5 = \log_{10} 1 - \log_{10}\text{H}_c$

$7.5 = 0 - \log_{10}\text{H}_c$

$\log_{10}\text{H}_c = -7.5$

$\text{H}_c = 10^{-7.5}$

$\doteq 3.162\times10^{-8}$

The hydrogen ion concentration is approximately 3.162×10^{-8} mol/L.

b.

$$\text{pH} = \log_{10} \frac{1}{\text{H}_c}$$

$$5.6 = \log_{10} \frac{1}{\text{H}_c}$$

$$5.6 = \log_{10} 1 - \log_{10} \text{H}_c$$

$$5.6 = 0 - \log_{10} \text{H}_c$$

$$\log_{10} \text{H}_c = -5.6$$

$$\text{H}_c = 10^{-5.6}$$

$$\doteq 2.512 \times 10^{-6}$$

The hydrogen ion concentration is about 2.512×10^{-6} mol/L.

6. a. 50 dB sound:

$$L_{\text{dB}} = 10 \log_{10} \frac{I_m}{I_r}$$

$$50 = 10 \log_{10} \frac{I_m}{I_r}$$

$$5 = \log_{10} \frac{I_m}{I_r}$$

$$10^5 = \frac{I_m}{I_r}$$

$$I_m = I_r \times 10^5$$

25 dB sound:

$$L_{\text{dB}} = 10 \log_{10} \frac{I_m}{I_r}$$

$$25 = 10 \log_{10} \frac{I_m}{I_r}$$

$$2.5 = \log_{10} \frac{I_m}{I_r}$$

$$10^{2.5} = \frac{I_m}{I_r}$$

$$I_m = I_r \times 10^{2.5}$$

$$\frac{I_m \text{ 50 dB sound}}{I_m \text{ 25 dB sound}} = \frac{I_r \left(10^5\right)}{I_r \left(10^{2.5}\right)}$$

$$= 10^{2.5}$$

$$\doteq 316$$

The 50 dB sound is about 316 times louder than the 25 dB sound.

b.

$$\frac{I_r \left(10^{10.8}\right)}{I_r \left(10^{9.4}\right)} = 10^{1.4}$$

$$\doteq 25$$

The lawnmower was approximately 25 times louder before it was repaired.

7. a. Earthquake A:

$$6.5 = \log_{10} \left(\frac{I_m}{I_r} \right)$$

$$10^{6.5} = \frac{I_m}{I_r}$$

$$I_m = 10^{6.5} I_r$$

Earthquake B:

$$4.7 = \log_{10} \left(\frac{I_m}{I_r} \right)$$

$$10^{4.7} = \frac{I_m}{I_r}$$

$$I_m = 10^{4.7} I_r$$

$$\frac{I_m \text{ Earthquake A}}{I_m \text{ Earthquake B}} = \frac{10^{6.5} I_r}{10^{4.7} I_r}$$
$$= 10^{6.5-4.7}$$
$$= 10^{1.8}$$
$$\doteq 63.1$$

Earthquake A is approximately 63.1 times greater than Earthquake B.

b. $$\frac{I_m \text{ Earthquake C}}{I_m \text{ Earthquake D}} = \frac{10^{5.9} I_r}{10^{5.1} I_r}$$
$$= 10^{5.9-5.1}$$
$$= 10^{0.8}$$
$$\doteq 6.3$$

Earthquake C is about 6.3 times greater than Earthquake D.

8. $$A = A_0 (3)^{\frac{t}{p}}$$
$$27\,000 = 2000(3)^{\frac{t}{4}} \quad \text{(Divide both sides by 2000.)}$$
$$13.5 = 3^{\frac{t}{4}}$$
$$\log 13.5 = \log 3^{\frac{t}{4}} \quad \text{(Take the log of both sides.)}$$
$$\frac{t}{4} = \frac{\log 13.5}{\log 3}$$
$$\frac{t}{4} \doteq \frac{1.130\,334}{0.477\,121}$$
$$t \doteq 9$$

It will take approximately 9 h.

Section 4: Follow-up Activities

Extra Help

1. Steps in solving $3^x = 7$ are as follows:

$\log 3^x = \log 7$ (Take base **10** log of each side.)
$x \log 3 = \log 7$ (**Power** Law)
$x = \frac{\log 7}{\log 3}$ (**Divide** both sides by $\log 3$.)
$\doteq$ **1.771** (to three decimal places)

2. a. The formula $\log_a b = \frac{\log_c b}{\log_c a}$ is called the **change of base** formula.

b. Use this formula to evaluate $\log_4 9$.

$$\log_4 9 = \frac{\log_{10} 9}{\log_{10} 4}$$

$$\log_4 9 \doteq 1.584\,96 \qquad \text{(to five decimal places)}$$

3. Steps in solving $x^{\frac{3}{7}} = 9$ are as follows:

$$\log x^{\frac{3}{7}} = \log 9$$

$$\frac{3}{7}\log x = \log 9 \qquad \text{(Power Law)}$$

$$\log x = \frac{7}{3}\log 9$$

$$\log x \doteq 2.226\,566 \qquad \text{(to six decimal places)}$$

$$x \doteq 168.4868 \qquad \text{(to four decimal places)}$$

4. Steps for solving $\log_3 5x - \log_3 (2x-3) = 2$ are as follows:

$$\log_3\left(\frac{5x}{2x-3}\right) = 2 \qquad \text{(Quotient Law)}$$

$$\frac{5x}{2x-3} = 3^2 \qquad \text{(exponential form)}$$

$$5x = 9(2x-3)$$

$$x = \frac{27}{13}$$

5. Common logarithms use base **10**.

6. The formula $G = G_0\left(2^{cy}\right)$ is used for predicting **population growth**.

7. The formula $a = a_0\left(2^{-\frac{t}{h}}\right)$ represents the amount remaining in the decaying process of radioactive substances. In the formula, h represents the **half-life** of the substance.

8. The formula $A = P\left(1+\frac{r}{n}\right)^{nt}$ is the **compound interest** formula.

9. In the formula $\text{pH} = \log_{10}\frac{1}{\text{H}_c}$, the H_c represents the **hydrogen ion concentration** in mol/L.

10. The loudness of sound is measured in **decibels**. The formula is $L = 10\log_{10}\frac{I_m}{I_r}$, where I_m is the intensity of sound being **measured**.

11. The **Richter** scale uses the formula $\log_{10}\frac{I_m}{I_r}$ to compare the intensities of **earthquakes**.

12. a.

$$3^{4x} = 5^{2x+1}$$

$$\log 3^{4x} = \log 5^{2x+1} \quad \text{(Take logs of both sides.)}$$

$$4x\log 3 = (2x+1)\log 5 \quad \text{(Power Law)}$$

$$4x\log 3 = 2x\log 5 + \log 5 \quad \text{(Distributive Law)}$$

$$4x\log 3 - 2x\log 5 = \log 5$$

$$x(4\log 3 - 2\log 5) = \log 5$$

$$x = \frac{\log 5}{4\log 3 - 2\log 5}$$

$$\doteq \frac{0.698\,970}{1.908\,485 - 1.397\,940}$$

$$\doteq 1.369 \quad \text{(to three decimal places)}$$

b.

$$\log_4 (3x-1) + \log_4 (x+1) = 1$$

$$\log_4 [(3x-1)(x+1)] = 1$$

$$\log_4 [(3x-1)(x+1)] = \log_4 4$$

$$(3x-1)(x+1) = 4$$

$$3x^2 + 2x - 1 = 4$$

$$3x^2 + 2x - 5 = 0$$

$$(3x+5)(x-1) = 0$$

$3x+5=0$ or $x-1=0$

$x = -\frac{5}{3}$ $\qquad$ $x = 1$

The value $-\frac{5}{3}$ is inadmissible; thus, $x = 1$.

c.

$$\log_3 2x - \log_3 (2x-1) = 2$$

$$\log_3 \frac{2x}{2x-1} = 2 \quad \text{(Quotient Law)}$$

$$\frac{2x}{2x-1} = 3^2 \quad \text{(exponential form)}$$

$$2x = 9(2x-1)$$

$$2x = 18x - 9$$

$$-16x = -9$$

$$x = \frac{9}{16}$$

13.

$$a = a_0\left(2^{-\frac{t}{h}}\right)$$

$$a = a_0\left(\frac{1}{2}\right)^{\frac{t}{h}} \qquad \left(2^{-\frac{t}{h}} = \frac{1}{2^{\frac{t}{h}}} = \frac{1}{(2)^{\frac{t}{h}}} = \left(\frac{1}{2}\right)^{\frac{t}{h}}\right)$$

$$4 = 40\left(\frac{1}{2}\right)^{\frac{t}{28}}$$

$$\frac{1}{10} = \left(\frac{1}{2}\right)^{\frac{t}{28}}$$

$$\log\frac{1}{10} = \log\left(\frac{1}{2}\right)^{\frac{t}{28}} \quad \text{(Take log of both sides.)}$$

$$\log\frac{1}{10} = \frac{t}{28}\log\frac{1}{2} \quad \text{(Power Law)}$$

$$t = \frac{28\log\frac{1}{10}}{\log\frac{1}{2}}$$

$$\doteq \frac{28(-1)}{-0.301\,030}$$

$$\doteq 93 \text{ years}$$

Enrichment

1. **a.** 2.0794 **b.** 3.2189
c. 2.4849 **d.** 0.6931
e. -0.2231 **f.** -5.2983

2. **a.** $\ln 6 \doteq 1.791\,759$
$e^{1.791\,759} \doteq 6$

b. $\ln 17 \doteq 2.833\,213$
$e^{2.833\,213} \doteq 17$

c. $\ln 0.3 \doteq -1.203\,973$
$e^{-1.203\,973} \doteq 0.3$

d. $\ln 0.008 \doteq -4.828\,314$
$e^{-4.828\,314} \doteq 0.008$

3. **a.** $e^x = 18$
$\ln e^x = \ln 18$
$x \ln e = \ln 18$
$x(1) = \ln 18$
$x \doteq 2.890\,372$

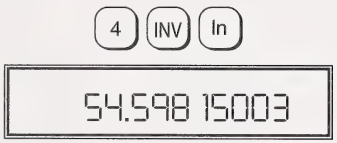
b. $e^x = 27$
$x \doteq 3.295\,837$

c. $-0.713\,350$

d. $-3.036\,554$

4. **a.** $\ln x = 4$

$\log_e x = 4$ is equivalent to $e^4 = x$

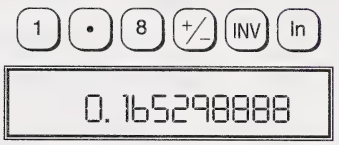


Thus, $x \doteq 54.598\,150\,03$.

b. 12.182 493 96

c. $\ln x = -1.8$

$\log_e x = -1.8$ is equivalent to $e^{-1.8} = x$



Thus, $x \doteq 0.165\,298\,888$.

d. 0.067 205 512

5. $\log_8 8\,657\,432 = \dfrac{\log 8\,657\,432}{\log 8}$

$\doteq \dfrac{6.937\,389\,089}{0.903\,089\,987}$

$\doteq 7.6818$

6. $\log_a 8 = \dfrac{\log_c 8}{\log_c a}$ (change of base)

$\log_a 8 = \dfrac{\log_8 8}{\log_8 a}$ (Let $c = 8$.)

$\therefore \dfrac{\log_8 8}{\log_8 a} = 7$ (given $\log_a 8 = 7$)

$\dfrac{1}{\log_8 a} = 7$

$\log_8 a = \dfrac{1}{7}$

7. a.

LS	RS
$(\log_c d)(\log_d c)$	1
$= \left(\dfrac{\log d}{\log c}\right)\left(\dfrac{\log c}{\log d}\right)$	
$= 1$	
LS =	RS

b. $\log_a b = \dfrac{1}{s^2}$ (1)

$\log_b a = 2s^4$ (2)

From the solution in 7.a., $\log_a (b) \times \log_b (a) = 1$. (3)

Substitute (1) and (2) into (3).

$\dfrac{1}{s^2} \times 2s^4 = 1$

$2s^2 = 1$

$s^2 = \dfrac{1}{2}$

$\therefore s = \pm\dfrac{1}{\sqrt{2}}$

$= \pm\dfrac{\sqrt{2}}{1}$

8. $\ln N = 2.303 \log N$

$\ln 15.7 \doteq 2.303 \log 15.7$

$\doteq 2.303 \times 1.195\,899\,652$

$\doteq 2.754$

9. a. $\ln \dfrac{1}{6} = \ln 1 - \ln 6$

$= \ln 1 - \ln(2 \times 3)$

$= \ln 1 - [\ln 2 + \ln 3]$

$\doteq 0 - [0.6931 + 1.0986]$

$\doteq -1.7917$

b. $\ln 12 = \ln(2 \times 2 \times 3)$

$= \ln 2 + \ln 2 + \ln 3$

$\doteq 0.6931 + 0.6931 + 1.0986$

$\doteq 2.4848$

10. Let $x = 6e^{-0.4}$.

$\ln x = \ln 6e^{-0.4}$ (Take the natural log of both sides.)

$\ln x = \ln 6 + (-0.4)\ln e$ (Product Law and Power Law)

$\ln x \doteq 1.791\,76 + (-0.4)1$ ($\log_e e = 1$)

$\ln x \doteq 1.391\,76$

$x \doteq 4.0219$

Thus, $6e^{-0.4} \doteq 4.0219$.

11. a. $P = 18.4e^{0.012t}$

For 1996, the time period t is sixteen years from 1980.

$P = 18.4e^{0.012(16)}$

$= 18.4e^{0.192}$

$\doteq 18.4(1.211\,671)$

$\doteq 22.29$

$\doteq 22.3$

In 1996 the population will be approximately 22.3 million.

b. $P = 18.4e^{0.012t}$

Substitute 25 for P then take the natural logarithm of each side.

$25 = 18.4e^{0.012t}$

$\ln 25 = \ln 18.4e^{0.012t}$

$\ln 25 = \ln 18.4 + 0.012t \ln e$

$\ln 25 = \ln 18.4 + 0.012t$

$$t = \frac{\ln 25 - \ln 18.4}{0.012}$$

$$\doteq \frac{3.218\,876 - 2.912\,351}{0.012}$$

$\doteq 25.54$

In 25.5 years from 1980, the population will be about 25 million.

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